# OccBin: A Toolkit for Solving Dynamic Models With Occasionally Binding Constraints Easily

Luca Guerrieri, Federal Reserve Board

and

Matteo Iacoviello, Federal Reserve Board

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# What We Do

- Inequality constraints that bind occasionally arise in a wide array of economic applications.
- We describe how to adapt a first-order perturbation approach and apply it in a piecewise fashion to handle occasionally binding constraints.
- We solve three examples of dynamic stochastic models with this approach:
    1. A real business cycle model with a constraint on investment;
    2. A new Keynesian model subject to the zero lower bound on the policy interest rate;
    3. A textbook example of optimal consumption choice in the presence of liquidity constraints.
- In each case, we compare the piecewise linear perturbation solution with a high-quality numerical solution that can be taken to be virtually exact.

# Contributions

1. We outline an algorithm to obtain a piecewise linear solution.
   - While the individual elements of the algorithm are not original, our recombination simplifies the application of this type of solution to a general class of models.
   - We have developed a MATLAB toolbox that extends Dynare.
2. We present a systematic assessment of the quality of the piecewise linear perturbation method relative to a virtually exact solution.
   - Where applicable, the virtually exact solution is obtained by dynamic programming on a very fine lattice for the state variables of the model.
   - In addition, following Christiano and Fisher (2000), we use spectral methods, which have been found to be highly accurate; for instance see Aruoba et al. (2006).

# The Solution Approach

- Because standard perturbation methods only provide a local approximation, they cannot capture occasionally binding constraints without adaptation.
- Our analysis builds on an insight that has been used extensively in the literature on the effects of attaining the zero-lower bound on nominal interest rates.
- Occasionally binding constraints can be handled as different regimes of the same model.
  - Under one regime, the occasionally binding constraint is slack.
  - Under the other regime, the same constraint is binding.
- The piecewise linear solution method involves linking the first-order approximation of the model around the same point under each regime.

# The Two Regimes

- Reference regime M1 (occasionally binding constraint is slack)
  Linearized system can be expressed as:

$$\mathcal{A} E_t X_{t+1} + \mathcal{B} X_t + \mathcal{C} X_{t-1} + \mathcal{E} \epsilon_t = 0, \tag{M1}$$

- Alternative regime M2 (occasionally binding constraint binds)
  Linearized system (around same non-stochastic steady state) can be expressed as:

$$\mathcal{A}^* E_t X_{t+1} + \mathcal{B}^* X_t + \mathcal{C}^* X_{t-1} + \mathcal{D}^* + \mathcal{E}^* \epsilon_t = 0. \tag{M2}$$

- Assume BK conditions hold in M1, and that absent shocks system is expected to return to M1 in finite time
- We are now in a position to define a solution for our model.

# Definition

## Definition

A solution for a model with an occasionally binding constraint is a function $f : X_{t-1} \times \epsilon_t \to X_t$ such that the conditions under system $(M1)$ or the system $(M2)$ hold, depending on the evaluation of the occasionally binding constraint.

- Alternatively, given initial conditions $X_0$ and the realization of a shock $\epsilon_1$, the function $f$ can be expressed as a set of matrices $\mathcal{P}_t$, a set of matrices $\mathcal{R}_t$, and a matrix $\mathcal{Q}_1$, such that:

$$X_1 = \mathcal{P}_1 X_0 + \mathcal{R}_1 + \mathcal{Q}_1 \epsilon_1, \quad (1)$$

$$X_t = \mathcal{P}_t X_{t-1} + \mathcal{R}_t \quad \forall t \in \{2, \infty\}. \quad (2)$$

- At each point in time the matrices $\mathcal{P}_t$, $\mathcal{Q}_t$, $\mathcal{R}_t$ are time varying, even if they are functions of $X_{t-1}$ and $\epsilon_1$ only.

# The algorithm

The algorithm employs a guess-and-verify approach.

1. We guess the periods in which each regime applies.
2. Second, we proceed to verify and, if necessary, update the initial guess.

Here are the details:

# The Algorithm (continued)

1. Assume that from period $T$ onwards $(M1)$ applies in perpetuity. Then for any $t \geq T$, using standard perturbation methods, one can characterize the linear approximation to the decision rule for $X_t$, given $X_{t-1}$, as:

$$X_t = \mathcal{P} X_{t-1} + \mathcal{Q}\epsilon_t, \quad \text{(M1DR)}$$

Then for any $t \geq T$, $\mathcal{P}_t = \mathcal{P}$, $\mathcal{R}_t = 0$.

# The Algorithm (continued)

2. Using $X_T = \mathcal{P} X_{T-1}$ and Equation $(M2)$, the solution in period T-1 will satisfy the following matrix equation:

$$\mathcal{A}^* \mathcal{P} X_{T-1} + \mathcal{B}^* X_{T-1} + \mathcal{C}^* X_{T-2} + \mathcal{D}^* = 0. \quad (3)$$

Solve the equation above for $X_{T-1}$ to obtain the decision rule for $X_{T-1}$, given $X_{T-2}$:

$$X_{T-1} = -\left(\mathcal{A}^* \mathcal{P} + \mathcal{B}^*\right)^{-1} \left(\mathcal{C}^* X_{T-2} + \mathcal{D}^*\right). \quad (4)$$

Accordingly, $\mathcal{P}_{T-1} = -\left(\mathcal{A}^* \mathcal{P} + \mathcal{B}^*\right)^{-1} \mathcal{C}^*$ and
$R_{T-1} = -\left(\mathcal{A}^* \mathcal{P} + \mathcal{B}^*\right)^{-1} \mathcal{D}^*$
Notice that the solution in T-1 combines elements from the reference and alternative regimes.
Continuing to substitute in this fashion, one can see that the "weights" depend on the duration of the regimes.

# The Algorithm (continued)

3. Using $X_{T-1} = \mathcal{P}_{T-1} X_{T-2} + \mathcal{R}_{T-1}$ and either $(M1)$ or $(M2)$, as implied by the current guess of regimes, solve for $X_{T-2}$ given $X_{T-3}$.

4. Iterate back in this fashion until $X_0$ is reached, applying either $(M1)$ or $(M2)$ at each iteration, as implied by the current guess of regimes.

5. Depending on whether regime $(M1)$ or $(M2)$ is guessed to apply in period 1, $\mathcal{Q}_1 = -\left(\mathcal{A}\mathcal{P}_2 + \mathcal{B}\right)^{-1}\mathcal{E}$, or $\mathcal{Q}_1 = -\left(\mathcal{A}^*\mathcal{P}_2 + \mathcal{B}^*\right)^{-1}\mathcal{E}^*$.

6. Using the guess for the solution obtained in steps 1. to 5., compute paths for $X$ to verify the current guess of regimes. If the guess is verified, stop. Otherwise, update the guess for when regimes $(M1)$ and $(M2)$ apply and return to step 1.

# Features of the Solution

- Importantly, the solution that the algorithm produces is not just linear.
- The solution is highly nonlinear
    - The dynamics in one of the two regimes may crucially depend on how long one expects to be in that regime.
    - In turn, how long one expects to be in that regime depends on the state vector.
    - This interaction produces the high nonlinearity.

# Advantages and Disadvantages

The piecewise linear solution inherits some disadvantages of a linear perturbation method:

- Just like any linear solution, it discards all information relative to the possibility of unforeseen future shocks;
- It does not capture precautionary behavior linked to the possibility that a constraint may become binding in the future, as a result of shocks yet unrealized.

But it also inherits some great advantages:

- It is computationally fast.
- It is applicable to models with a large number of state variables even when the curse of dimensionality renders other higher-quality methods inapplicable.
- It is general and application of our algorithm to different models requires only minimal programming.

# Application 1: A Simple Asset Pricing Model

Consider the following asset pricing model

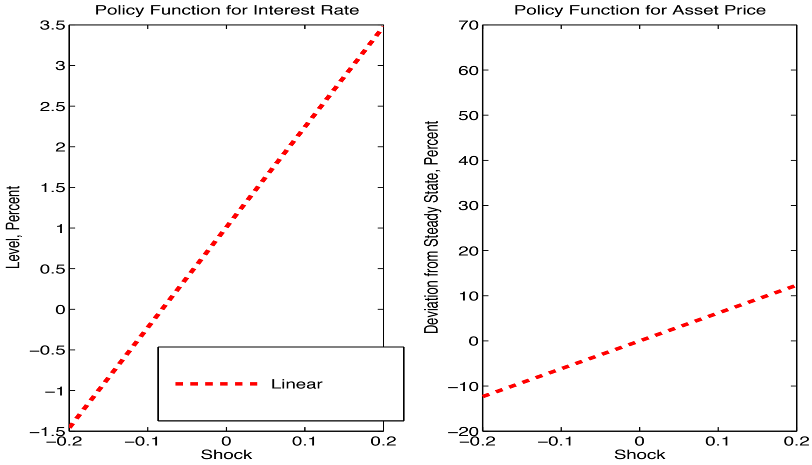
$$\begin{aligned} q_t &= \beta(1-\rho)E_t q_{t+1} + \rho q_{t-1} - \sigma r_t + u_t \\ r_t &= \max(\underline{r}, \phi q_t) \end{aligned}$$

$\beta$=0.99, $\rho$=0.5, $\phi$=0.5, $\underline{r}$=-0.01 $\sigma$=5
$u_t$ AR(1) process with $\rho_u$=0.5 and $\sigma_u$=0.05

For realizations of $u_t$ above a threshold, higher values of $u_t$ lead to higher asset prices and, through the feedback rule, higher interest rates (and there is no difference with linearized solution).

For realizations of $u_t$ below threshold, lower values of $u_t$ lead to much lower asset prices, since interest rates are bounded below and cannot offset the decline in $q_t$.

# Policy Functions for Simple Asset Pricing Model



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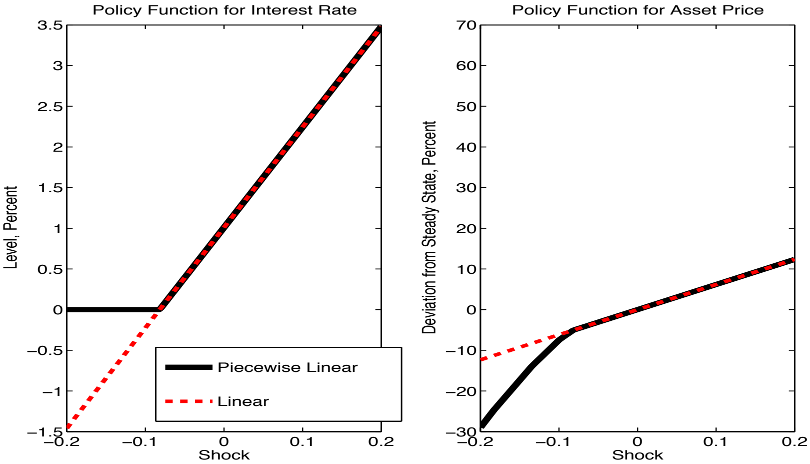

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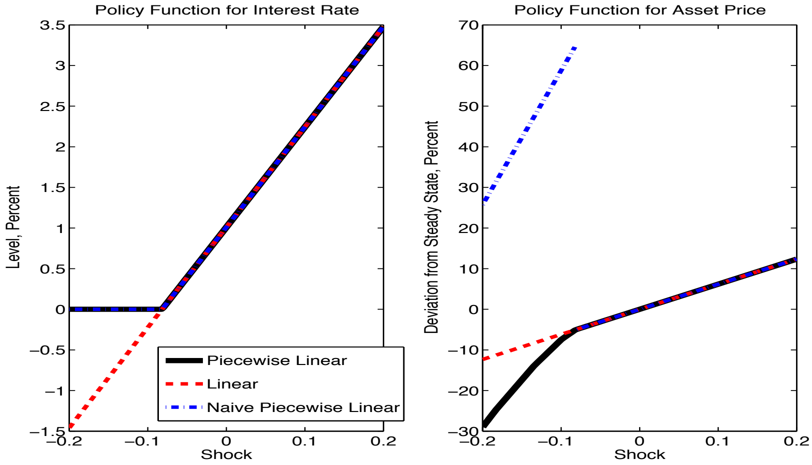

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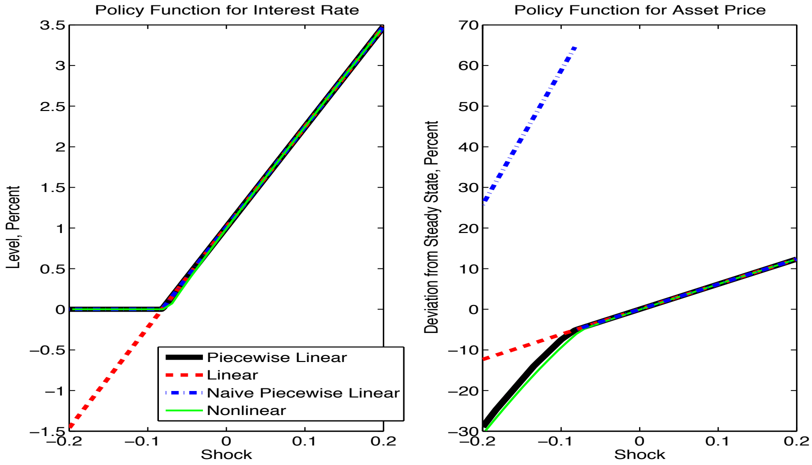

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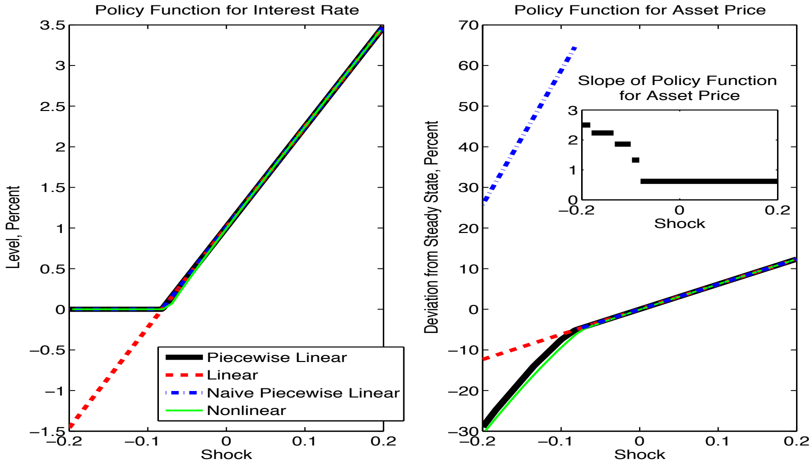


# Application 2 – RBC with constraint on investment

The planner maximizes households' utility

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma},$$

subject to the constraints:

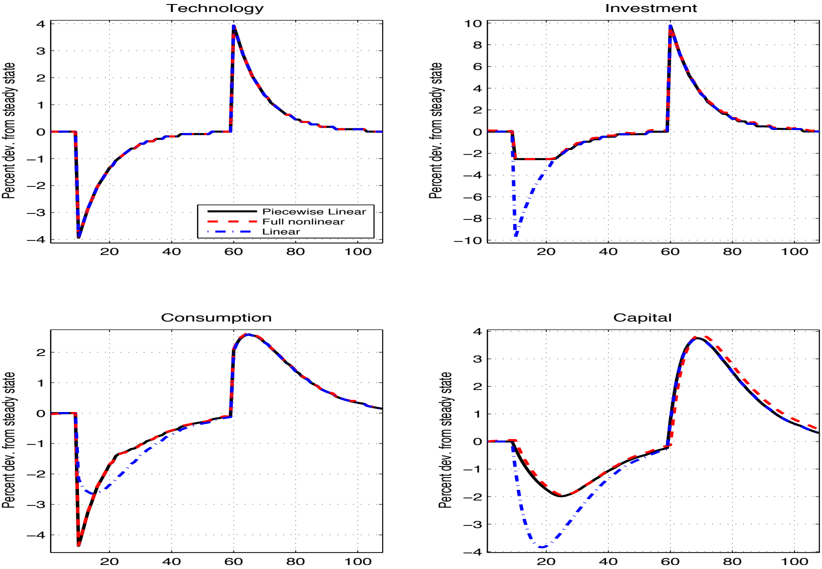
$$C_t + I_t = A_t K_{t-1}^{\alpha},$$
$$K_t = (1-\delta) K_{t-1} + I_t,$$
$$I_t \geq \phi I_{SS}.$$

The stochastic process for the technology $A_t$ is given by

$$\ln A_t = \rho \ln A_{t-1} + \sigma \epsilon_t.$$

Set $\gamma$=2. Set $\phi = 0.975$, implying that constraint binds about 40% of the time

# The response of shocks to technology




Units on the abscissae denote years.

# Euler Equation Errors

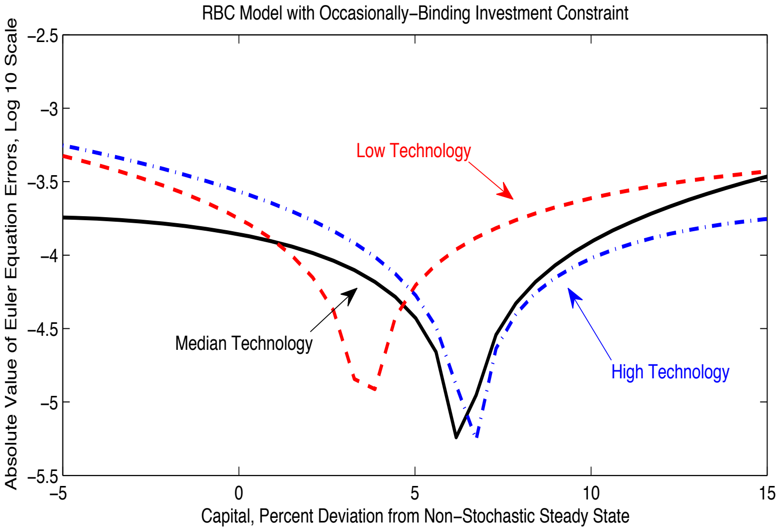
See the paper for comparisons of distributions, particular moments, and welfare cost of adopting a piecewise linear solution relative to a nonlinear, virtually exact solution.
Focus here on Euler errors:

$$C_t^{-\gamma} - \lambda_t = \beta E_t \left( C_{t+1}^{-\gamma} \left( 1 - \delta + \alpha A_{t+1} K_t^{\alpha-1} \right) - (1-\delta)\lambda_t \right).$$

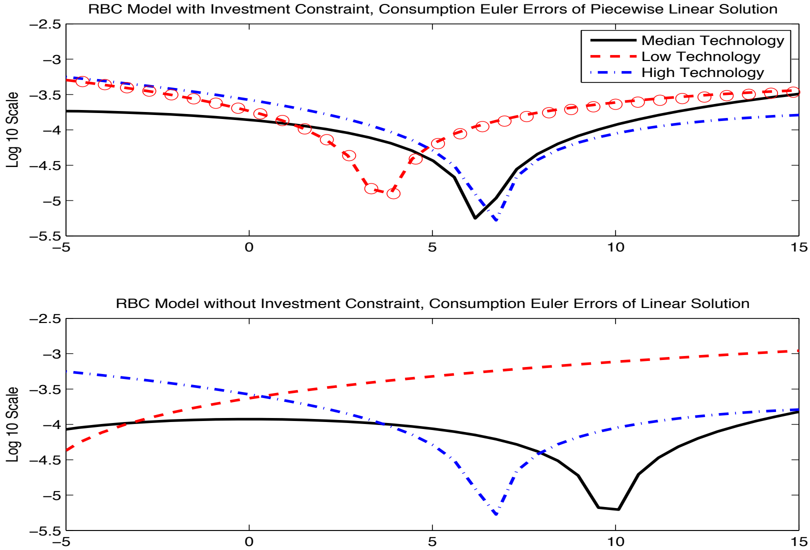
We define the Euler equation error as:

$$err_t = \frac{-C_t + \left\{ \lambda_t + E_t \beta \left[ C_{t+1}^{-\gamma} \left( (1-\delta) + \alpha A_{t+1} K_t^{\alpha-1} \right) - (1-\delta)\lambda_{t+1} \right] \right\}^{-\frac{1}{\gamma}}}{C_t}.$$

# Euler Errors for Our Solution Algorithm

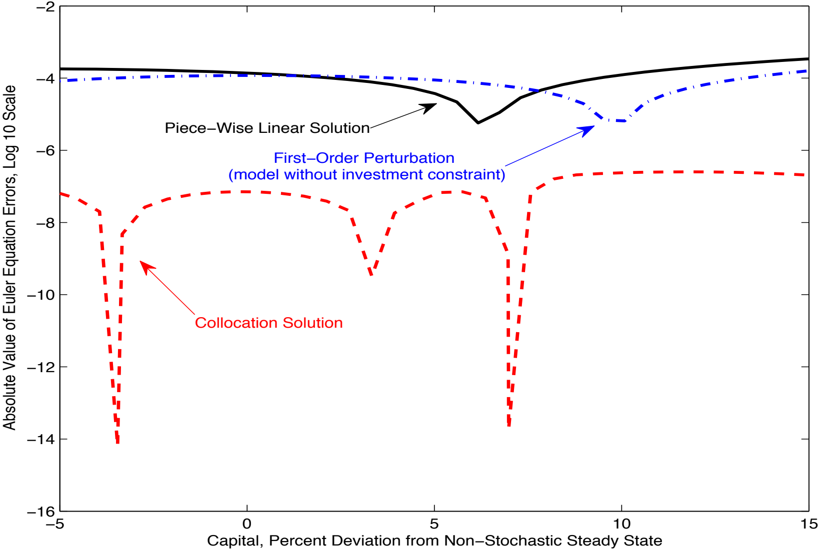


# Euler Errors: Model with vs Model w/o Constraint




..................... Capital (percent deviation from steady state) .................

# Euler Errors: Comparison with Full Nonlinear

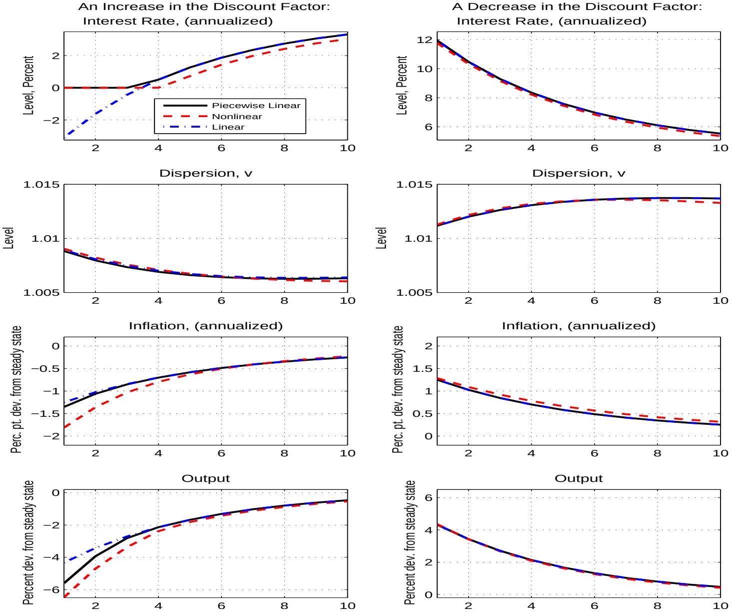


# Other Checks and Assessment

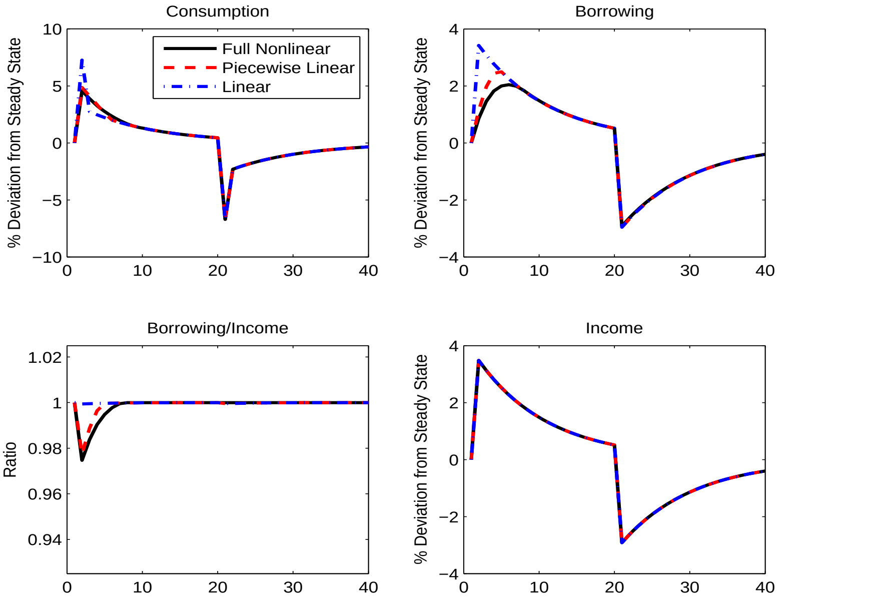
For each of the models considered, in the paper we also present:

- Comparisons of the distributions of individual endogenous variables;
- Comparisons of first and second moments;
- Frequency and duration of regime switching;
- A second measure of bounded rationality – a compensating subsidy for switching from the use of nonlinear decision rules to piecewise linear rules.
- In most cases, we are pleased with the performance of our algorithm.

# Impulse Responses: Calvo Model with ZLB



# Impulse Responses: Model with Borrowing Limit



# Estimation: Forming a Likelihood Function

- Can we use this solution technique when we are interested in estimating a model with occasionally binding constraints?
- Rewrite the solution to make explicit the relationship between reduced-form parameters and shocks:

$$X_t = \mathbf{P}(X_{t-1}, \epsilon_t)X_{t-1} + \mathbf{D}(X_{t-1}, \epsilon_t) + \mathbf{Q}(X_{t-1}, \epsilon_t)\epsilon_t$$

- This representation of the solution makes clear the basic endogeneity issue to be resolved to form a likelihood function.
- The standard Kalman filter allows for exogenous, but not endogenous variation in the reduced form coefficients.
- Viable options are simulation based filters, such as the Unscented Kalman filter or the Particle filter.
- Alternatively, consider an approach to forming the likelihood function that relies on a change in variables.

# Estimation: Forming a Likelihood Function

- The solution of the model takes the form:

$$X_t = \mathbf{P}(X_{t-1}, \epsilon_t) X_{t-1} + \mathbf{D}(X_{t-1}, \epsilon_t) + \mathbf{Q}(X_{t-1}, \epsilon_t)\epsilon_t$$

- ... and in terms of observables $Y_t$ , through the observation equation $Y_t = \mathbf{H} X_t$, we have:

$$Y_t = \mathbf{HP}(X_{t-1}, \epsilon_t) X_{t-1} + \mathbf{HD}(X_{t-1}, \epsilon_t) + \mathbf{HQ}(X_{t-1}, \epsilon_t)\epsilon_t$$

One can initialize $X_0$, and can recursively solve for $\epsilon_t$, given $X_{t-1}$ and the current realization of $Y_t$ .

# Estimation: Forming a Likelihood Function

- The standard change in variables argument requires forming the Jacobian of the inverse transformation.
- This Jacobian is notoriously costly to construct for numerical solutions.
- The local linearity implied by the piece-wise linear perturbation approach implies that this Jacobian is a byproduct of the solution.
- Given that $\epsilon_t$ is $NID(0, \boldsymbol{\Sigma})$, applying the change in variables argument implies that the log likelihood for $Y$ can be written as:

$$\log L = -\frac{T}{2}\log(\det(\boldsymbol{\Sigma})) - \frac{1}{2}\sum_t^T \hat{\epsilon}_t' \boldsymbol{\Sigma}^{-1}\hat{\epsilon}_t - \sum_t^T \log(|\det(\mathbf{H}\mathbf{Q}_t)|)$$

- Given parameters, solving, filtering and evaluating L takes seconds with serial processing in Matlab.

# Estimation Example

- Can the estimation approach suggested in the previous slide work in practice?
- In a related paper, we show that the answer is yes.
- We estimate a variant of the Smets-Wouters model extended to include a housing sector following Iacoviello (2005).
- We estimate the model subject to an occasionally binding constraint on housing wealth and subject to the zero lower bound on interest rates.
- The estimated model is described in "Collateral Constraints and Macroeconomic Asymmetries," available on Matteo's research page.

# Conclusions

- The piecewise perturbation approach retains key properties of the standard linear perturbation approach while extending the range of models that can be solved.
- The quality of the solution is comparable to that of the linear perturbation method for models that exclude occasionally binding constraints.
- The key advantages the piece-wise perturbation approach are:
  - It can be deployed with minimal adaptation;
  - It is applicable to models with a large number of state variables;
  - It is computationally fast.
- Our codes are available on Matteo's research page. Try them out.