1 Solving a Linear Approximation of the Real Business Cycle model with Fixed Labor Supply

Note: The material in these notes is adapted from a handout that Jonathan Heathcote prepared for a graduate class at Georgetown University.

1.1 Model description

Households seek to maximize utility given by:

$$\sum_{t=0}^{\infty} E_t \beta^t log(c_t).$$

Households have access to a production technology given by:

$$y_t = e^{z_t} k_t^{\alpha},$$

where z_t is a shock process governed by:

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \tag{1}$$

where ϵ_{t+1} is normally and independently distributed with mean 0 and variance σ^2. The law of motion for capital is

$$k_{t+1} = (1 - \delta) k_t + i_t.$$

Finally, the resource constraint for the economy implies that

$$c_t + i_t = y_t.$$

1.2 Necessary conditions for an equilibrium

To find the necessary conditions for an equilibrium setup the households maximization problem using the following Lagrangian:

$$\max_{c_t, K_{t+1}, i_t, \lambda_t, \gamma_t} \quad L = \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right. \tag{2}$$

$$+\beta^t \lambda_t \left[c_t + i_t - e^{z_t} k_t^{\alpha} \right] \tag{3}$$

$$\left. +\beta^t \gamma_t \left[(1 - \delta) k_t + i_t - k_{t+1} \right] \right\} \tag{4}$$

N.B.: the way in which you write the lagrangian constraints affects the interpretation of the multiplier, but does not affect the final solution.

The first-order conditions of the Lagrangian with respect to the maximization objects above are given by:

$$\frac{\partial L}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \tag{5}$$

$$\frac{\partial L}{\partial k_{t+1}} = \beta E_t \left[\lambda_{t+1} \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1}\right] - \gamma_t + \beta E_t \left[\gamma_{t+1}(1-\delta)\right] = 0 \tag{6}$$

$$\frac{\partial L}{\partial i_t} = \lambda_t - \gamma_t = 0 \tag{7}$$

$$\frac{\partial L}{\partial \lambda_t} = e^{z_t} k_t^{\alpha} - c_t - i_t = 0 \tag{8}$$

$$\frac{\partial L}{\partial \gamma_t} = (1-\delta)k_t + i_t - k_{t+1} = 0 \tag{9}$$

The necessary conditions for an equilibrium of the model are given by all the first-order conditions above, plus the shock process in equation (1).

1.3 Some manual intervention

Before attempting to solve the model, realize that the conditions for an equilibrium listed above can be reduced to a smaller set of equations. This manual intervention is simple in the case of this model, but might be substantially more involved for other models. Later, we are going to learn how to deploy some numerical techniques to avoid it altogether.

Using equation (7) and equation (5) notice that

$$\gamma_t = \frac{1}{c_t} \tag{10}$$

Substituting γ_t from equation (10) into equation (6) and collecting terms, we obtain:

$$E_t \beta \left[\frac{1}{c_{t+1}} \left(1 - \delta + \alpha e^{z_{t+1}} k_{t+1}^{(\alpha-1)}\right)\right] = \frac{1}{c_t}. \tag{11}$$

Solving the resource constraint in equation (8) for i_t and substituting in equation (9) one obtains

$$k_{t+1} = (1-\delta)k_t + e^{z_t} k_t^{\alpha} - c_t. \tag{12}$$

We have now expressed the necessary conditions for an equilibrium as three equations (11), (12), and (1) and three variables k,c, and z.

1.4 Model calibration and non-stochastic steady states

Before we can compute the steady state values of k, c, and z, we need to choose numerical values for the parameters in the model. Let $\delta = 0.025$, $\beta = 0.99$, $\alpha = 0.33$, and $\rho = 0.95$.

Let "*" denote steady state values. From equation (1), $z^* = 0$. Working on equation (11)

$$\beta \left[\frac{1}{c^*} (1 - \delta + \alpha k^{*\alpha-1}) \right] = \frac{1}{c^*}. \tag{13}$$

This implies that

$$k^* = \left[\frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta \right) \right]^{\frac{1}{\alpha-1}} \tag{14}$$

Finally, from equation(12), $k^* = (1-\delta)k^* + k^{*\alpha} - c^*$, which yields

$$c^* = k^{*\alpha} - \delta k^*. \tag{15}$$

Using the parameter choices above, the numerical steady states are: $z^* = 0$, $k^* \approx 28.3$, $c^* \approx 2.3$.

1.5 Linearizing the model

Let a "^" denote a variable's deviation from its non-stochastic steady state value, i.e., $\hat{c}_t = c_t - c^*$. Using the first-order Taylor series expansion around the non-stochastic steady state, one can express the necessary conditions for the model's equilibrium as:

$$-\frac{\beta}{c^{*2}}(1 - \delta + \alpha k^{*\alpha-1}) E_t \hat{c}_{t+1} + \frac{\beta}{c^*} \alpha(\alpha-1) k^{*(\alpha-2)} E_t \hat{k}_{t+1} + \frac{\beta}{c^*} \alpha k^{*\alpha-1} E_t \hat{z}_{t+1} = -\frac{1}{c^{*2}} \hat{c}_t \tag{16}$$

$$\hat{k}_{t+1} = -\hat{c}_t + (1 - \delta + \alpha k^{*\alpha-1}) \hat{k}_t + k^{*\alpha} \hat{z}_t \tag{17}$$

$$\hat{z}_{t+1} = \rho \hat{z}_t + \hat{\epsilon}_{t+1} \tag{18}$$

Next, deal with the conditional expectation operator. Start with $E_t c_t + 1$. Consider the identity: $E_t c_{t+1} = c_{t+1} + (E_t c_{t+1} - c_{t+1})$. This is simply saying that current expectation of next period's consumption can be expressed as the realization of next period's consumption plus an expectational error. Trivially, this relationship will also hold in deviation from steady state: $E_t\hat{c}_{t+1} = \hat{c}_{t+1} + (E_t\hat{c}_{t+1} - \hat{c}_{t+1})$. Now, introduce a little more notation. Let

$$\hat{\omega}_{ct+1} = (E_t\hat{c}_{t+1} - \hat{c}_{t+1}), \tag{19}$$

$$\hat{\omega}_{kt+1} = (E_t\hat{k}_{t+1} - \hat{k}_{t+1}), \tag{20}$$

$$\hat{\omega}_{zt+1} = (E_t\hat{z}_{t+1} - \hat{z}_{t+1}). \tag{21}$$

Solve definitions (19) to (21) for $E_t\hat{c}_{t+1}$, $E_t\hat{k}_{t+1}$, and $E_t\hat{z}_{t+1}$, respectively and substitute into equation (16). Then, one obtains:

$$\begin{aligned}
&-\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1})[\hat{c}_{t+1}+\hat{\omega}_{ct+1}] \\
&+\frac{\beta}{c^{*}}\alpha(\alpha-1)k^{*(\alpha-2)}[\hat{k}_{t+1}+\hat{\omega_{kt+1}}] \\
&+\frac{\beta}{c^{*}}\alpha k^{*\alpha-1}[\hat{z}_{t+1}+\hat{\omega}_{zt+1}] = -\frac{1}{c^{*2}}\hat{c}_t
\end{aligned} \tag{22}$$

Rewrite equations (22), (17), (18) in matrix form:

$$\psi\hat{x}_{t+1} + J\hat{\omega}_{t+1} = \phi\hat{x}_t + \hat{e}_{t+1}, \tag{23}$$

where

$$\hat{x}_s = \begin{pmatrix} \hat{c}_s \\ \hat{k}_s \\ \hat{z}_s \end{pmatrix}, \quad \hat{\omega}_s = \begin{pmatrix} \hat{\omega}_{cs} \\ \hat{\omega}_{ks} \\ \hat{\omega}_{zs} \end{pmatrix}, \quad \hat{e}_s = \begin{pmatrix} 0 \\ 0 \\ \hat{\epsilon}_s \end{pmatrix}$$

and where

$$\psi = \begin{pmatrix} -\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1}) & \frac{\beta}{c^{*}}\alpha(\alpha-1)k^{*(\alpha-2)} & \frac{\beta}{c^{*}}\alpha k^{*\alpha-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} -\frac{1}{c^{*2}} & 0 & 0 \\ -1 & (1-\delta+\alpha k^{*\alpha-1}) & k^{*\alpha} \\ 0 & 0 & \rho \end{pmatrix}.$$

Equation (23) can be simplified a little more, so as to yield:

$$\psi \hat{x}_{t+1} = \phi \hat{x}_t + \hat{f}_{t+1} \quad with \;\; \hat{f}_{t+1} = \begin{pmatrix} -\psi_1 \hat{\omega}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix} \tag{24}$$

From here onwards, we need to proceed numerically.

1.6 Case 1: ψ is invertible

The simplifications we brought to the necessary conditions for an equilibrium have ensured that ψ is invertible. We will tackle what to do when ψ is not invertible later on.

Premultiplying equation (24) by ψ^{-1}, one can see that:

$$\hat{x}_{t+1} = \psi^{-1} \phi \hat{x}_t + \psi^{-1} \hat{f}_{t+1} \tag{25}$$

You might be tempted to stop here, but you'd not be quite done yet. Remember that $\hat{f}_{t+1}$ is a function of expectational errors. We need to solve for those. We'll get there in what might initially appear a rather circuitous route.

Let $A = \psi^{-1}\phi$. The next step is to find the eigenvalues and eigenvectors of A. Place the eigenvalues of A along the diagonal of the matrix D. Arrange the corresponding eigenvectors of A along the columns of the matrix V. Fortunately you don't have to do this by hand. Matlab provides the function *eig* to construct the matrices V and D. Premultiplying equation (25) by V^{-1}, thus

$$V^{-1}\hat{x}_{t+1} = V^{-1} A \hat{x}_t + V^{-1} \psi^{-1} \hat{f}_{t+1}$$

Remember that $AV = VD$. Thus, $V^{-1}AV = D$. But then, $V^{-1}A = DV^{-1}$. Accordingly:

$$V^{-1}\hat{x}_{t+1} = DV^{-1}\hat{x}_t + V^{-1}\psi^{-1}\hat{f}_{t+1} \tag{26}$$

Next, change variables, let $\hat{y}_s = V^{-1}\hat{x}_s$. Accordingly, we can rewrite the equation (26) above as:

$$\hat{y}_{t+1} = D\hat{y}_t + V^{-1}\psi^{-1}\hat{f}_{t+1} \tag{27}$$

This last transformation puts us in a really good position. We have written the necessary conditions for an equilibrium in our model so that each condition involves only one variable (albeit, a linear combination of the original variables) and some linear combination of the expectational errors and the innovation to technology. Let the diagonal entries of D be denoted by d_i. Furthermore, let $\eta = V^{-1}\psi^{-1}$. Accordingly, each equation can be written as:

$$\hat{y}_{it+1} = d_i \hat{y}_{it} + \eta_{row(i)} \hat{f}_{t+1}. \tag{28}$$

Notice that if $|d_i| > 1$, then taking the conditional expectation at time t and iterating on (28) implies that $E_t \hat{y}_{is}$ might eventually explode, i.e. $\lim_{s\to\infty} |E_t \hat{y}_{is}| = \infty$ under some conditions. When $|d_i| > 1$, the d_i eigenvalue is said to be explosive.

To see the argument more clearly:

$$E_t \hat{y}_{it+1} = d_i E_t \hat{y}_{it} + E_t \eta_{row(i)} \hat{f}_{t+1}, \tag{29}$$

but by rational expectations the current expectation of future expectational errors is 0. Thus, $E_t \eta_{row(i)} \hat{f}_{t+1} = 0$ and

$$E_t \hat{y}_{it+1} = d_i \hat{y}_{it}. \tag{30}$$

Iterating forward

$$\frac{1}{d_i} E_t \hat{y}_{it+2} = \hat{y}_{it+1} \tag{31}$$

and substituting into (30), one can see that

$$E_t E_{t+1} \hat{y}_{it+2} = d_i^2 \hat{y}_{it}. \tag{32}$$

Using the law of iterated expectations:

$$E_t \hat{y}_{it+2} = d_i^2 \hat{y}_{it}. \tag{33}$$

Iterating forward some more, the power on the term d_i will keep growing, which is the basis for the claim of explosiveness above when $\hat{y}_{it}$ is nonzero.

Now, suppose we impose a non-explosiveness condition of the form $\lim_{s\to\infty} E_t y_i s = 0$. The interpretation of this condition is that far enough in the future, we should expect that all variables converge in expectation to their non-stochastic steady state values. If $|d_i| > 1$, this condition can only be satisfied if $\hat{y}_{it}$ is 0 at all times.

Given our parametric choices, we can confirm that, for the model we are considering, we indeed have one explosive eigenvalue (and 2 non-explosive eigenvalues), which ensures that there will be a unique rational expectation equilibrium in the neighborhood of the non-stochastic steady state (see Blanchard and Kahn). Then, we know that for some i, $\hat{y}_{it} = 0 \ \forall \ t$. This can buy us two interesting simplifications.

Simplification # 1

How about the expectational errors? Given that $y_{it} = 0 \ \forall \ t$, from equation (28), we have that

$$\eta_{row(i)} \hat{f}_{t+1} = 0.$$

Remembering the definition of $\hat{f}$, the above implies

$$\eta_{row(i)} \begin{pmatrix} -\psi_1 \hat{\omega}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix} = 0, \tag{34}$$

which can be used to solve for $\psi_1 \hat{\omega}_{t+1}$. Thus,

$$\psi_1 \hat{\omega}_{t+1} = \frac{\eta_{i,3}}{\eta_{i,1}} \hat{\epsilon}_{t+1}. \tag{35}$$

The above equation has some interesting economic interpretation. The expectational errors are linearly related to the innovation to the productivity shock process $\hat{\epsilon}_{t+1}$.

At this point we could declare victory. For any set of predetermined conditions $\hat{x}_t = (\hat{c}_t, \hat{k}_t, \hat{z}_t)'$, and innovation $\hat{\epsilon}_t$, we can use V^{-1} to map those conditions into a vector $\hat{y}_t$. Using the restriction in equation (35), we can construct $\hat{f}_{t+1}$. Then, using equation (27) we can obtain $\hat{y}_{t+1}$, which can be transformed back into $\hat{x}_{t+1}$ using V.

Simplification # 2

While we could have stopped at simplification #1, a little more algebra will yield even more rewards.

Since $\hat{y}_{is} = (V^{-1})_{row(i)}\hat{x}_s$, imposing the non-explosiveness condition, we have that:

$$y_{it} = \sum_j (V^{-1})_{ij} = (V^{-1})_{i1}c_t + (V^{-1})_{i2}k_t + (V^{-1})_{i3}z_t = 0 \tag{36}$$

We can use this relationship to solve for one of the variables in terms of the other two. Pick $\hat{c}_t$. Thus,

$$\hat{c}_t = \begin{pmatrix} c_k & c_z \end{pmatrix} \begin{pmatrix} k_t \\ z_t \end{pmatrix}, \tag{37}$$

where $c_k = -\frac{(V^{-1})_{i2}}{(V^{-1})_{i1}}$ and $c_z = -\frac{(V^{-1})_{i3}}{(V^{-1})_{i1}}$. In general, for each explosive eigenvalue, we can solve one variable out of the system.

Finally, we can substitute the restrictions we have found for $\hat{c}_t$ and $\psi_1\hat{\omega}_{t+1}$ into the the original system of equations in matrix form. Focus on the second and third equation in (25). They can be rewritten as:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} = \begin{pmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{z}_t \end{pmatrix} + \begin{pmatrix} \psi^{-1}_{row(2)} \\ \psi^{-1}_{row(3)} \end{pmatrix} \begin{pmatrix} -\frac{\eta_{i,3}}{\eta_{i,1}}\hat{\epsilon}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}.$$

Substituting equation (37) and (35), into the above:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} = \left(\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} \begin{pmatrix} c_k & c_z \end{pmatrix} + \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{z}_t \end{pmatrix} \right) + \begin{pmatrix} \psi^{-1}_{row(2)} \\ \psi^{-1}_{row(3)} \end{pmatrix} \begin{pmatrix} -\frac{\eta_{i,3}}{\eta_{i,1}}\hat{\epsilon}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}.$$

Remember that the last two rows of ψ had ones along the diagonal and zeros everywhere else. That's going to carry through to its inverse, ψ^{-1}. So, $(\psi^{-1})_{row(2)} = \{0, 1, 0\}$, and $(\psi^{-1})_{row(3)} = \{0, 0, 1\}$, which implies:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} = \left(\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} \begin{pmatrix} c_k & c_z \end{pmatrix} + \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{z}_t \end{pmatrix} \right) + \begin{pmatrix} 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}.$$

1.7 Case 2: ψ is singular

The algebraic manipulations we performed onto the system of necessary conditions for an equilibrium ensured that ψ in equation (23) would be invertible. In fact, if we had not substituted all the identities out of the system, ψ would not have been invertible. Ensuring the invertibility of ψ is too laborious a task in all but the simplest models we are interested in solving. Fortunately, we can deploy some more matrix algebra to help us.

To fix ideas, let's work with a very simple example of a setup in which ψ is not going to be invertible. Consider again the basic RBC model described earlier, but this time, rather than using the resource constraint $y_t = c_t + i_t$ to substitute out i_t from the capital accumulation equation $k_{t+1} = (1-\delta)k_t + i_t$, ignore this simplification. Upon linearization, the necessary conditions for an equilibrium in our model can then be written as:

$$-\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1})E_t\hat{c}_{t+1} + \frac{\beta}{c^*}\alpha(\alpha-1)k^{*(\alpha-2)}E_t\hat{k}_{t+1} + \frac{\beta}{c^*}\alpha k^{*\alpha-1}E_t\hat{z}_{t+1} = -\frac{1}{c^{*2}}\hat{c}_t \quad (38)$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t \quad (39)$$

$$0 = \hat{c}_t - \alpha k^{*\alpha-1}\hat{k}_t + \hat{i}_t - k^{*\alpha}\hat{z}_t \quad (40)$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \hat{\epsilon}_{t+1} \quad (41)$$

Proceeding analogously to our earlier case, rewrite equations (38) to (41) in matrix form, so as to yield:

$$\phi\hat{x}_{t+1} = \psi\hat{x}_t + \hat{f}_{t+1}.$$

Departing slightly from the earlier method, this time, if we are particular about the order in which we arrange equations and variables, it will have a big payoff later. The reasons, will become apparent in a little while.

We want to order equations and variables according to these rules:

1. Predetermined variables (e.g., values for shocks, capital stocks) come first, non predetermined variables (e.g., consumption) second.

2. Inter-temporal equations (e.g. law of motion for capital, shock processes) are placed above, intratemporal equations below.

3. The inter-temporal equations that have expectational errors, such as the Euler equation for consumption, are placed last among the set of intertemporal equations.

One ordering of the necessary conditions for an equilibrium for the model under study that satisfies the rules above is: (41), (39), (38), (40). One ordering for the variables in the system that satisfies the rules above is $\hat{z}_s, \hat{k}_s, \hat{c}_s, \hat{i}_t$.

Adopting the ordering above for equations and variables, we obtain:

$$\hat{x}_s = \begin{pmatrix} \hat{z}_t \\ \hat{k}_t \\ \hat{c}_t \\ \hat{i}_t \end{pmatrix}, \quad \hat{f}_s = \begin{pmatrix} \hat{\epsilon}_s \\ 0 \\ -\psi_1 \hat{\omega}_s \\ 0 \end{pmatrix} \quad \hat{\omega}_s = \begin{pmatrix} \hat{\omega}_{zs} \\ \hat{\omega}_{ks} \\ \hat{\omega}_{cs} \\ \hat{\omega}_{is} \end{pmatrix},$$

$$\psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\beta}{c^*}\alpha k^{*\alpha-1} & \frac{\beta}{c^*}\alpha(\alpha-1)k^{*(\alpha-2)} & -\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\phi = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 1 \\ 0 & 0 & -\frac{1}{c^{*2}} & 0 \\ -k^{*\alpha} & -\alpha k*^{\alpha-1} & 1 & 1 \end{pmatrix}.$$

Inspecting ψ, the last row of zeros (or the last column) make it singular.

Fortunately, we can still proceed much in the same way as when ψ is invertible, but we just need to be a little more patient with the algebra. For starters, use the generalized complex Schur decomposition on the matrices ϕ and ψ. If you have never used this decomposition before, don't panic. It is still a cause of great happiness to me that Matlab can perform this task for us with the command *qz*. The generalized complex Schur decomposition can be applied to any pair of square matrices ϕ and ψ. It yields matrices $Q, Z, \phi_\triangle, \psi_\triangle$ that have some interesting properties:

$$
\begin{aligned}
Q\phi Z &= \phi_\triangle \\
Q\psi Z &= \psi_\triangle \\
Q^H Q = I, &\quad Z^H Z = I,
\end{aligned}
$$

where, "^{H}" denotes the transpose of the complex conjugate, and I the identity matrix. One of the beautiful properties of this decomposition is that $\phi_\triangle$ and $\psi_\triangle$ are (upper) triangular matrices. Furthermore, the generalized eigenvalues of $\phi_\triangle$ and $\psi_\triangle$ are the same as those of the original matrices ϕ and ψ. In fact, the generalized eigenvalues can simply be found by dividing one by one the diagonal entries of $\phi_\triangle$ by the diagonal entries of $\psi_\triangle$.

The Schur decomposition is not unique. We want to reorder it so that the stable eigenvalues are associated with the upper rows of $\phi_\triangle$ and $\psi_\triangle$, and the unstable eigenvalues with the lower rows. This can be done with the Matlab command *ordqz*.

Let's go back to our system of conditions for an equilibrium in matrix form $\phi\hat{x}_{t+1} = \psi\hat{x}_t + \hat{f}_{t+1}$. We can now rewrite it as:

$$
Q^H\phi_\triangle Z^H\hat{x}_{t+1} = Q^H\psi_\triangle Z^H\hat{x}_t + \hat{f}_{t+1}. \tag{42}
$$

Premultiplying through by Q,

$$
\phi_\triangle Z^H\hat{x}_{t+1} = \psi_\triangle Z^H\hat{x}_t + Q\hat{f}_{t+1}. \tag{43}
$$

As you might predict, this invites a natural change in variables. Let $\hat{y}_s = Z^H\hat{x}_s$. Then,

$$
\phi_\triangle\hat{y}_{t+1} = \psi_\triangle\hat{y}_t + Q\hat{f}_{t+1}. \tag{44}
$$

Next let's partition $\hat{y}_s$ into two parts $\hat{y}_s = \begin{pmatrix} \hat{y}_{Ss} \\ \hat{y}_{Us} \end{pmatrix}$. The partition $\hat{y}_{Ss}$ has n_S rows, as many as the number of stable generalized eigenvalues of ϕ and ψ. The partition $\hat{y}_{Us}$ has n_U rows, as many as the number of unstable generalized eigenvalues. Applying the same partitioning to equation (44), one obtains:

$$
\begin{pmatrix} \psi_{\triangle 11} & \psi_{\triangle 12} \\ \psi_{\triangle 21} & \psi_{\triangle 22} \end{pmatrix} \begin{pmatrix} \hat{y}_{St+1} \\ \hat{y}_{Ut+1} \end{pmatrix} = \begin{pmatrix} \phi_{\triangle 11} & \phi_{\triangle 12} \\ \phi_{\triangle 21} & \phi_{\triangle 22} \end{pmatrix} \begin{pmatrix} \hat{y}_{St} \\ \hat{y}_{Ut} \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ \hat{f}_{Ut+1} \end{pmatrix} \tag{45}
$$

Thus, for example, $\psi_{\triangle 11}$ has dimensions $n_S \times n_S$.

Since all the entries of $\psi_{\triangle 21}$ and of $\phi_{\triangle 21}$ are zeros, the non-explosiveness condition implies that $\hat{y}_{Us} = 0$ for all s. To see this, notice that $\phi_{\triangle 22} V = \psi_{\triangle} V D$, where V collects the generalized eigenvectors of $\phi_{\triangle 22}$ and $\psi_{\triangle 22}$, and D is a diagonal matrix whose non-zero entries are the generalized eigenvalues. By construction, the generalized eigenvalues of $\phi_{\triangle 22}$ and $\psi_{\triangle 22}$ are unstable. So, we can construct a similar argument to the one we used in the previous section, in the case of ψ invertible, to show that the non-explosiveness condition does imply $\hat{y}_{Us} = 0$ for all s.

Note that the change in variables we used above also implies that $\hat{x}_s = Z\hat{y}_s$. Partititioning the system leads to:

$$\begin{pmatrix} \hat{x}_{St} \\ \hat{x}_{Ut} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \hat{y}_{St} \\ \hat{y}_{Ut} \end{pmatrix}$$

But, since $\hat{y}_s = 0$ $\forall s$, we have that $\hat{x}_{St} = Z_{11}\hat{y}_{St}$, and also that $\hat{x}_{Ut} = Z_{21}\hat{y}_{Ut}$. Assuming that Z_{11} is invertible, $\hat{y}_{St} = (Z_{11})^{-1}\hat{x}_{St}$. But then,

$$\hat{x}_{Ut+1} = Z_{21} Z_{11}^{-1} \hat{x}_{St+1} \tag{46}$$

Notice that, by construction, if the Blanchard-Kahn conditions are satisfied, $\hat{x}_{Ut}$ only holds jump variables, and $\hat{x}_{St}$ only holds non-jump predetermined variables. So, equation (46) gives us a way do deduce all the jump variables from the predetermined variables, without any additional recourse to the innovations to the shock processes other than their effects already embedded in $\hat{x}_{St+1}$.

Finally, we need to solve for $\hat{x}_{St+1}$. From equation (45), using the result from the non-explosiveness condition that $\hat{y}_s = 0$ $\forall$ s, we have that

$$\psi_{\triangle 11}\hat{y}_{St+1} = \phi_{\triangle 11}\hat{y}_{St} + \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ \hat{f}_{Ut+1} \end{pmatrix} \tag{47}$$

But the non-explosiveness condition also implies

$$\begin{pmatrix} Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ \hat{f}_{Ut+1} \end{pmatrix} = 0$$

Assuming that Q_{22} is invertible, we get that

$$\hat{f}_{Ut+1} = -Q_{22}^{-1} Q_{21} \hat{f}_{St+1} \tag{48}$$

Notice that, by construction, $\hat{f}_{St+1}$ will hold innovations to shocks and no expectational errors. Conversely, $\hat{f}_{St+1}$ will hold expectational errors, and no innovations. Accordingly, equation (48) gives us a way to retrieve expectational errors from fundamental innovations. The Blanchard-Kahn conditions for uniqueness, given our ordering rules, will ensure invertibility of Q_{22}.

Substituting equation (48) into equation (47), and remembering that $\hat{y}_{St} = Z_{11}^{-1}\hat{x}_{St}$, one obtains

$$\psi_{\triangle 11}(Z_{11})^{-1}\hat{x}_{St+1} = \phi_{\triangle 11}(Z_{11})^{-1}\hat{x}_{St} + \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ -Q_{22}^{-1} Q_{21} \hat{f}_{St+1} \end{pmatrix}$$

Premultiplying by $Z_{11}\psi_{\triangle 11}^{-1}$ and collecting terms, then we can rewrite the equation above as

$$\hat{x}_{St+1} = Z_{11}\psi_{\triangle 11}^{-1}\phi_{\triangle 11} Z_{11}^{-1}\hat{x}_{St} + Z_{11}^{-1}\psi_{\triangle 11}\left(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}\right)\hat{f}_{St+1}. \tag{49}$$

Equations (49) and (46) give us a way to solve for all variables of interest. In fact, with just one more set of painless substitutions, they can be rewritten in a more visually appealing way:

$$\hat{x}_{St+1} = A\hat{x}_{St} + B\hat{f}_{St+1} \tag{50}$$

$$\hat{x}_{Ut+1} = C\hat{x}_{St+1} \tag{51}$$

$$A = Z_{11}\psi_{\triangle 11}^{-1}\phi_{\triangle 11}(Z_{11})^{-1} \tag{52}$$

$$B = Z_{11}\psi_{\triangle 11}^{-1}\left(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}\right) \tag{53}$$

$$C = Z_{21}(Z_{11})^{-1}, \tag{54}$$

which completes the "A,B,C" of model solving.