Contents

1 Handout 5

The aim of these notes is to show you a way to incorporate nominal rigidities in the DGE models we developed in the first few classes. We'll start with the closed economy setup. Much of what follows should look familiar from previous courses, as well as the classes we have had so far. A secondary purpose of the notes below is to introduce notation and fix ideas that we shall exploit again when we consider the open economy setup.

2 The Household Problem

There is a continuum of households of measure one. Households maximize utility subject to their budget constraint, and the law of motion for capital

$$\max_{[C_t(h),W_t(h),Mb_t(h),I_t(h),K_{t+1}(h),B_{t+1}(h)]} E_t \sum_{j=0}^{\infty} \beta^j \left(U(C_{t+j}(h), C_{t+j-1}(h)) \right.$$
$$\left. V(L_{t+j}(h)) + \nu(\frac{Mb_t(h)}{P_t}) \right) + \beta^j \lambda_{t+j}(h) \left[\Pi_t(h) + T_{t+j}(h) + W_{t+j}(h) L_{t+j}(h) \right.$$
$$+R_{k,t+j} K_{t+j}(h) - \frac{1}{2}\psi_k P_{t+j} K_{t+j}(h) \left(\frac{I_{t+j}(h)}{K_{t+j}(h)} - \delta \right)^2 - \frac{1}{2}\psi_I P_{t+j} \frac{\left(I_{t+j}(h) - I_{t+j-1}(h)\right)^2}{I_{t+j-1}(h)}$$
$$\left. -P_{t+j} C_{t+j}(h) - P_{t+j} I_{t+j}(h) - \int_s \psi_{t+j+1,t+j} B_{t+j+1}(h) + B_{t+j}(h) - Mb_{t+j}(h) + Mb_{t+j-1}(h) \right]$$
$$+\beta^j Q_{t+j}(h) \left[(1-\delta) K_{t+j}(h) + I_{t+j}(h) - K_{t+j+1}(h) \right],$$

and subject to the labor demand schedule $L_t(h) = L_t \left(\frac{W_t(h)}{W_t} \right)^{-\frac{1+\theta_w}{\theta_w}}$

Notice that the budget constraint above is written in nominal terms. Furthermore, notice that we have introduced adjustment costs for both capital and investment. We shall explore the differences betweem these types of adjustment costs in the homework problems.

2.1 First-Order Condition for Investment

$$\begin{aligned} \frac{\partial}{\partial I_t(h)} &= -\lambda_t(h)\psi_k P_t K_t(h) \left(\frac{I_t(h)}{K_t(h)} - \delta \right) \frac{1}{K_t(h)} - \lambda_t(h) P_t \psi_I \frac{(I_t(h) - I_{t-1}(h))}{I_{t-1}(h)} \\ &- \lambda_t(h) P_t + Q_t(h) + \beta\lambda_{t+1}(h)\psi_I P_{t+1} \frac{(I_{t+1}(h) - I_t(h))}{I_t(h)} \\ &+ \frac{1}{2}\beta\lambda_{t+1}(h)\psi_I P_{t+1} \frac{(I_{t+1}(h) - I_t(h))^2}{I_t(h)^2} = 0 \end{aligned} \tag{1}$$

collecting terms and solving for $Q_t(h)$

$$\begin{aligned} Q_t(h) &= \lambda_t(h) P_t + \lambda_t(h)\psi_k P_t \left(\frac{I_t(h)}{K_t(h)} - \delta \right) + \lambda_t(h) P_t \psi_I \left(\frac{I_t(h)}{I_{t-1}(h)} - 1 \right) \\ &- \beta\lambda_{t+1}(h)\psi_I P_{t+1} \left(\frac{I_{t+1}(h)}{I_t(h)} - 1 \right) - \frac{1}{2}\beta\lambda_{t+1}(h)\psi_I P_{t+1} \left(\frac{I_{t+1}(h)}{I_t(h)} - 1 \right)^2 \end{aligned} \tag{2}$$

Define $\Lambda_t(h) = \lambda_t(h)P_t$, $q_t(h) = \frac{Q_t(h)}{\lambda_t(h)P_t}$. Substitute these definitions in the above equation, and divide by $\Lambda_t(h)$

$$\begin{aligned} q_t(h) &= 1 + \psi_k \left(\frac{I_t(h)}{K_t(h)} - \delta \right) + \psi_I \left(\frac{I_t(h)}{I_{t-1}(h)} - 1 \right) - \psi_I \beta \frac{\Lambda_{t+1}(h)}{\Lambda_t(h)} \left(\frac{I_{t+1}}{I_t} - 1 \right) - \\ & \frac{1}{2} \psi_I \beta \frac{\Lambda_{t+1}}{\Lambda_t} \left(\frac{I_{t+1}}{I_t} - 1 \right)^2 \end{aligned} \tag{3}$$

2.2 First-Order Condition for Capital

$$\begin{aligned} \frac{\partial}{\partial K_{t+1}(h)} &= \beta \lambda_{t+1}(h) \left[R_{k,t+1} - \frac{1}{2} \psi_K P_{t+1} \left(\frac{I_{t+1}(h)}{K_{t+1}(h)} - \delta \right)^2 \right. \\ &+ \left. \psi_K P_t K_{t+1}(h) \left(\frac{I_{t+1}(h)}{K_{t+1}(h)} - \delta \right) \frac{I_{t+1}(h)}{(K_{t+1}(h))^2} \right] - Q_t(h) + \beta(1-\delta) Q_{t+1}(h) = 0 \end{aligned} \tag{4}$$

Solve for $Q_t(h)$ and collect terms

$$\begin{aligned} Q_t(h) &= \beta(1-\delta) Q_{t+1}(h) + \beta P_{t+1} \lambda_{t+1}(h) \left[\frac{R_{K,t+1}}{P_{t+1}} - \frac{1}{2} \psi_K \left(\frac{I_{t+1}(h)}{K_{t+1}(h)} - \delta \right)^2 \right. \\ &+ \left. \psi_K \frac{I_{t+1}(h)}{K_{t+1}(h)} \left(\frac{I_{t+1}(h)}{K_{t+1}(h)} - \delta \right) \right] \end{aligned} \tag{5}$$

Define $\Lambda_t(h) = \lambda_t(h)P_t$, $q_t(h) = \frac{Q_t(h)}{\lambda_t(h)P_t}$, $r_{k,t} = \frac{R_{k,t}}{P_t}$. Substitute these definitions in the above equation, and divide by $\Lambda_t(h)$

$$\begin{aligned} q_t(h) &= \beta \frac{\Lambda_{t+1}(h)}{\Lambda_t(h)} \left[(1-\delta) q_{t+1}(h) + r_{k,t+1} - \frac{1}{2} \psi_k \left(\frac{I_{t+1}(h)}{K_{t+1}(h)} - \delta \right)^2 \right. \\ & \left. + \psi_k \frac{I_{t+1}(h)}{K_{t+1}(h)} \left(\frac{I_{t+1}(h)}{K_{t+1}(h)} - \delta \right) \right] \end{aligned} \tag{6}$$

2.3 First-Order Condition for Bond Holding

Taking the first order condition for bond holding and reintroducing the notation for state dependence yields

$$\lambda_{s(t)} \psi_{s(t+1),s(t)} - \beta \lambda_{s(t+1)} Prob(s(t+1)|s(t)) = 0 \tag{7}$$

Rearranging

$$\psi_{s(t+1),s(t)} = \beta\frac{\lambda_{s(t+1)}}{\lambda_{s(t)}}Prob(s(t+1)|s(t)) \tag{8}$$

Defining i_t as the risk free interest rate yields

$$\frac{1}{1+i_t} = \int_{s(t+1),s(t)} \psi_{s(t+1),s(t)}ds(t+1) = \int_{s(t+1),s(t)} \beta\frac{\lambda_{s(t+1)}}{\lambda_{s(t)}}Prob(s(t+1)|s(t))ds(t+1), \tag{9}$$

which is equivalent to

$$\frac{1}{1+i_t} = E_t\beta\frac{\lambda_{t+1}}{\lambda_t} = E_t\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{P_t}{P_{t+1}}. \tag{10}$$

Notice that $E_t\frac{\frac{P_{t+1}}{P_t}}{1+i_t} = E_t\frac{\pi_{t+1}}{1+i_t}$is dubbed the "ex-ante real rate."

2.4 First-Order Condition for the Wage Rate

Let $1-\xi_w$ be the probability that in any period $t+j$ a worker h can reset $W_{t+j}(h)$. If the wage is not reset, then it is updated according to $W_{t+j}(h) = W_t(h)\pi^j$, where π is the target inflation rate.

$$\begin{aligned}\frac{\partial}{\partial W_t(h)} &= \sum_{j=0}^{\infty}(\xi_w\beta)^j V_L\left(L_{t+j}(h)\right)\frac{\partial L_{t+j}(h)}{\partial W_t(h)} + \\ &\quad + (\xi_w\beta)^j \lambda_{t+j}(h)(1+\tau_w)\pi^j\left[W_t(h)\frac{\partial L_{t+j}(h)}{\partial W_t(h)} + L_{t+j}(h)\right] = 0\end{aligned} \tag{11}$$

Rearranging terms, and multiplying both sides of the equation above by $W_t(h)$, one obtains:

$$\begin{aligned}&- \sum_{j=0}^{\infty}(\xi_w\beta)^j V_L\left(L_{t+j}(h)\right)\frac{\frac{\partial L_{t+j}(h)}{\partial W_t(h)}}{\frac{L_{t+j}(h)}{W_t(h)}}L_{t+j}(h) = \\ &\sum_{j=0}^{\infty}(\xi_w\beta)^j (1+\tau_w)\lambda_{t+j}(h)\pi^j L_{t+j}(h)W_t(h)\left[1+\frac{\frac{\partial L_{t+j}(h)}{\partial W_t(h)}}{\frac{L_{t+j}(h)}{W_t(h)}}\right]\end{aligned} \tag{12}$$

From the labor demand schedule $\frac{\frac{\partial L_t(h)}{\partial W_t(h)}}{\frac{L_t(h)}{W_t(h)}}$, the wage elasticity of labor supply, is equal to $-\frac{1+\theta_w}{\theta_w}$. Similarly $\frac{\frac{\partial L_{t+j}(h)}{\partial W_t(h)}}{\frac{L_{t+j}(h)}{W_t(h)}}$, given $W_{t+j}(h) = \pi^j W_t(h)$, is also equal to $-\frac{1+\theta_w}{\theta_w}$. Substituting this in the above equation, and multiplying through by θ_w

$$\sum_{j=0}^{\infty}(\xi_w\beta)^j V_L\left(L_{t+j}(h)\right)(1+\theta_w)L_{t+j}(h) = -\sum_{j=0}^{\infty}(\xi_w\beta)^j (1+\tau_w)\pi^j\lambda_{t+j}(h)L_{t+j}(h)W_t(h) \tag{13}$$

Remember that $\Lambda_t = P_t \lambda_t$. Rearranging terms

$$\sum_{j=0}^{\infty} (\beta\xi_w)^j \left[(1+\tau_w) \frac{W_t(h)}{W_t} \frac{(W_t \pi^j)}{W_{t+j}} \frac{W_{t+j}}{P_{t+j}} \Lambda_{t+j} + (1+\theta_w) V_L \left(L_{t+j}(h)\right) \right] L_{t+j}(h) = 0 \tag{14}$$

Let $\zeta_t = \frac{W_t}{P_t}$. Let $w_{r,t} = \frac{W_t(h)}{W_t}$. Let $\omega_{t+j} = \frac{W_{t+j}}{W_{t+j-1}\pi}$.

$$\sum_{j=0}^{\infty} (\beta\xi_w)^j \left[(1+\tau_w) w_{r,t} \left(\prod_{s=1}^{j} \frac{1}{\omega_{t+s}} \right) \zeta_{c,t+j} \Lambda_{t+j} + (1+\theta_w) V_L \left(L_{t+j}(h)\right) \right] L_{t+j}(h) = 0 \tag{15}$$

Assume that the utility function is such that $V(L) = \frac{\chi_0}{1-\chi}(1-L)^{1-\chi}$

$$\begin{aligned} &\sum_{j=0}^{\infty} (\beta\xi_w)^j \, (1+\tau_w) w_{r,t} \left(\prod_{s=1}^{j} \frac{1}{\omega_{t+s}} \right) \zeta_{t+j} \Lambda_{t+j} L_{t+j}(h) = \\ &\sum_{j=0}^{\infty} (\beta\xi_w)^j \, (1+\theta_w) \chi_0 \left(1 - L_t(h)\right)^{-\chi} L_{t+j}(h) \end{aligned} \tag{16}$$

Log Linearizing:

$$\begin{aligned} &(1+\tau_w)\zeta\Lambda L \sum_{j=0}^{\infty} (\beta\xi_w)^j \left[\hat{w}_{r,t} + \hat{\zeta}_{t+j} + \hat{\Lambda}_{t+j} + \hat{L}_{t+j}(h) \right] + \\ &-(1+\tau_w)\zeta\Lambda_c L \sum_{j=0}^{\infty} (\beta\xi_w)^j \sum_{s=1}^{j} \hat{\omega}_{t+s} = \\ &(1+\theta_w)\chi_0 L(1-L)^{-\chi} \sum_{j=0}^{\infty} (\beta\xi_w)^j \left[\hat{L}_{t+j}(h) + \frac{L\chi}{1-L} \hat{L}_{t+j}(h) \right] \end{aligned} \tag{17}$$

Notice that in steady state $(1+\tau_w)\zeta\Lambda L = (1+\theta_w)\chi_0 L(1-L)^{-\chi}$. Also, notice that both sides of the equation have a $\hat{L}_{t+j}(h)$ term that can be cancelled.

$$\begin{aligned} &\sum_{j=0}^{\infty} (\beta\xi_w)^j \left[\hat{w}_{r,t} + \hat{\zeta}_{t+j} + \hat{\Lambda}_{t+j} \right] - \sum_{j=0}^{\infty} (\beta\xi_w)^j \sum_{s=1}^{j} \hat{\omega}_{t+s} = \\ &\sum_{j=0}^{\infty} (\beta\xi_w)^j \left[\frac{L\chi}{1-L} \hat{L}_{t+j}(h) \right] \end{aligned} \tag{18}$$

Assuming that the wage of worker h has not been reset j periods hence, one can rewrite the labor demand schedule as follows: $L^d_{t+j} \left(\frac{w_t(h)}{w_t} \frac{w_t \pi^j}{w_{t+j}} \right)^{-\frac{1+\theta_w}{\theta_w}}$. Log linearizing this equation, one obtains $\hat{L}_{t+j}(h) = \hat{L^d}_{t+j} - \frac{1+\theta_w}{\theta_w} \left(\hat{w}_{r,t} - \sum_{s=1}^{j} \hat{\omega}_{t+s} \right)$. Incorporating the linearized labor demand schedule in the equation above, and pulling the $\hat{w}_{r,t}$ term out of the summation, one obtains:

$$\begin{aligned} &\frac{1}{1-\beta\xi_w} \hat{w}_{r,t} + \sum_{j=0}^{\infty} (\beta\xi_w)^j \left(\hat{\zeta}_{t+j} + \hat{\Lambda}_{t+j} \right) - \sum_{j=1}^{\infty} (\beta\xi_w)^j \sum_{s=1}^{j} \hat{\omega}_{t+s} = \\ &\sum_{j=0}^{\infty} (\beta\xi_w)^j \left[\frac{L\chi}{1-L} \left(\hat{L}_{t+j} - \frac{1+\theta_w}{\theta_w} \hat{w}_{r,t} + \frac{1+\theta_w}{\theta_w} \sum_{s=1}^{j} \hat{\omega}_{t+s} \right) \right] \end{aligned} \tag{19}$$

Collect the ω_{t+s} terms, and multiply through by -1:

$$\sum_{j=0}^{\infty} (\beta\xi_w)^j \left[-\hat{\zeta}_{t+j} - \hat{\Lambda}_{t+j} + \frac{L\chi}{1-L}\hat{L}_{t+j}\right] = \tag{20}$$
$$\left(1 + \frac{L\chi}{1-L}\frac{1+\theta_w}{\theta_w}\right)\left[\frac{1}{1-\beta\xi_w}\tilde{w}_{r,t} - \sum_{j=1}^{\infty}(\beta\xi_w)^j \sum_{s=1}^{j}\hat{\omega}_{t+s}\right]$$

Let $\hat{\mu}_t = \frac{1}{\left(1+\frac{L\chi}{1-L}\frac{1+\theta_w}{\theta_w}\right)}\left[-\hat{\zeta}_{t+j} - \hat{\Lambda}_{t+j} + \frac{L\chi}{1-L}\hat{L}_{t+j}\right]$, then

$$\sum_{j=0}^{\infty} (\beta\xi_w)^j \hat{\mu}_{t+j} = \frac{1}{1-\beta\xi_w}\hat{w}_{r,t} - \sum_{j=1}^{\infty}(\beta\xi_w)^j \sum_{s=1}^{j}\hat{\omega}_{t+s} \tag{21}$$

Pull out the first term from each of the two summations

$$\hat{\mu}_t + \beta\xi_w \sum_{j=0}^{\infty} (\beta\xi_w)^j \hat{\mu}_{t+j+1} = \frac{1}{1-\beta\xi_w}\hat{w}_{r,t} - \frac{\beta\xi_w}{1-\beta\xi_w}\hat{\omega}_{t+1} - \beta\xi_w \sum_{j=0}^{\infty}(\beta\xi_w)^j \sum_{s=1}^{j}\hat{\omega}_{t+s+1} \tag{22}$$

Take the lead of the equation we had before the above one and multiply both sides by $\beta\xi_w$, then

$$\beta\xi_w \sum_{j=0}^{\infty} (\beta\xi_w)^j \hat{\mu}_{t+j+1} = \frac{\beta\xi_w}{1-\beta\xi_w}\hat{w}_{r,t+1} - \beta\xi_w \sum_{j=0}^{\infty}(\beta\xi_w)^j \sum_{s=1}^{j}\hat{\omega}_{t+s+1} \tag{23}$$

Subtracting equation above from the one before it, one can see that

$$\hat{\mu}_t + \frac{\beta\xi_w}{1-\beta\xi_w}\hat{w}_{r,t+1} + \frac{\beta\xi_w}{1-\beta\xi_w}\hat{\omega}_{t+1} = \frac{1}{1-\beta\xi_w}\hat{w}_{r,t} \tag{24}$$

The aggregate wage can be written as: $\hat{w}_t = \xi_w \hat{w}_{t-1} + (1-\xi)\hat{w}_t(h)$. Realizing that $\hat{w}_{r,t} = \hat{w}_t(h) - \hat{w}_t$ and that $\hat{\omega}_t = \hat{w}_t - \hat{w}_{t-1}$, one can see that $\hat{w}_{r,t} = \frac{\xi_w}{1-\xi_w}\hat{\omega}_t$. Substituting in the equation above:

$$\hat{\mu}_t + \frac{\beta\xi_w}{1-\beta\xi_w}\frac{\xi_w}{1-\xi_w}\hat{\omega}_{t+1} + \frac{\beta\xi_w}{1-\beta\xi_w}\hat{\omega}_{t+1} = \frac{1}{1-\beta\xi_w}\frac{\xi_w}{1-\xi_w}\hat{\omega}_t \tag{25}$$

Collecting terms

$$\frac{\beta\xi_w}{(1-\beta\xi_w)(1-\xi_w)}\hat{\omega}_{t+1} + \hat{\mu}_t = \frac{\xi_w}{(1-\beta\xi_w)(1-\xi_w)}\hat{\omega}_t \tag{26}$$

$$\hat{\omega}_t = \beta\hat{\omega}_{t+1} + \kappa_w \hat{\mu}_t \tag{27}$$

where $\kappa_w = \frac{(1-\beta\xi_w)(1-\xi_w)}{\xi_w}$.

2.5 First-Order Condition for Consumption

$$\frac{\partial}{\partial c_t(h)} = \frac{\partial U\left(c_t(h), c_{t-1}(h)\right)}{\partial c_t(h)} + \beta \frac{\partial U\left(c_{t+1}(h), c_t(h)\right)}{\partial c_t(h)} - \lambda_t(h) P_t = 0 \tag{28}$$

Assume $U(c_t, c_{t-1}) = \frac{c_t^{1-\sigma}}{1-\sigma} - \psi_c \frac{1}{2} \frac{(c_t - c_{t-1})^2}{c_{t-1}}$. Then

$$c_t(h)^{-\sigma} - \psi_c \frac{c_t(h) - c_{t-1}(h)}{c_{t-1}(h)} + \beta \left(\psi_c \frac{c_{t+1}(h) - c_t(h)}{c_t(h)} + \frac{1}{2} \psi \frac{(c_{t+1}(h) - c_t(h))^2}{c_t^2(h)} \right) = \lambda_t(h) P_t \tag{29}$$

Remember that $\Lambda_{c,t} = \lambda_t P_t$. Then,

$$c_t(h)^{-\sigma} - \psi_c \frac{c_t(h) - c_{t-1}(h)}{c_{t-1}(h)} + \beta \left(\psi_c \frac{c_{t+1}(h) - c_t(h)}{c_t(h)} + \frac{1}{2} \psi \frac{(c_{t+1}(h) - c_t(h))^2}{c_t^2(h)} \right) = \Lambda_t(h) \tag{30}$$

3 The Production Sector

3.1 Final Producers

Competitive final producers aggregate intermediate products for resale. Their production function is

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{1}{1+\theta_p}} \right]^{1+\theta_p} \tag{31}$$

From the zero profit condition

$$P_t = \left[\int_0^1 P_t(f)^{-\frac{1}{\theta_p}} \right]^{-\theta_p}. \tag{32}$$

From their cost minimization problem, we can see that aggregate demand for each intermediate product $Y_t(f)$ is given by

$$Y_t(f) = \left[\frac{P_t(f)}{P_{q,t}} \right]^{-\frac{1+\theta_p}{\theta_p}} Y_t. \tag{33}$$

3.2 Intermediate Producers

Intermediate firms are monopolistically competitive. There is complete mobility of capital and labor across firms. Intermediate firms take input prices as given. $L_t^d(f)$, which enters the intermediate firms' production function is an aggregate over the skills supplied by each household, and takes the form $L_t^d(f) = \left(\int_h L_t^d(h)^{\frac{1}{1+\theta_w}}\right)^{1+\theta_w}$.

3.3 The Demand for an Individual Household's Labor

We can think of breaking down the production process so that a perfectly competitive aggregator with the same preferences for household skills as the intermediate firms aggregates the various skills of individual households and rents this product out to firms.

The profit maximization problem of this aggregator can be written as

$$\min_{L_t(h)} \int_h W_t(h)L_t(h) + W_t\left[L_t - \left(\int_h L_t(h)^{\frac{1}{1+\theta_w}}\right)^{1+\theta_w}\right] \tag{34}$$

The solution of this problem yields:

$$L_t(h) = \left(\frac{W_t(h)}{W_t}\right)^{-\frac{1+\theta_w}{\theta_w}} L_t \tag{35}$$

Furthermore, the zero-profit condition on the aggregator implies:

$$W_t = \left(\int_h W_t(h)^{-\frac{1}{\theta_w}}\right)^{-\theta_w} \tag{36}$$

3.4 Cost Minimization Problem

The cost function for intermediate firms is obtained by choosing capital and labor, subject to the production function, so as to minimize:

$$\min_{[K_t(f),L_t(f)]} R_{K,t}K_t(f) + W_tL_t^d(f) + \Sigma_t(f)\left(Y_t(f) - A_tK_t(f)^{\alpha}L_t^d(f)^{1-\alpha}\right). \tag{37}$$

Notice that $\Sigma_t(f)$ can be interpreted as the marginal cost for firm f. Given competition, and complete mobility of inputs, $\Sigma_t(f) = \Sigma_t$ for all f. Define $\zeta_t = \frac{w_t}{P_t}$, $r_{kt} = \frac{R_{kt}}{P_t}$, and $\sigma_t = \frac{\Sigma_t}{P_t}$. Then, from the above minimization problem

$$\sigma_t = \frac{r_{k,t}}{\alpha\frac{Y_t(f)}{K_t(f)}}, \tag{38}$$

and similarly

$$\sigma_t = \frac{\zeta_t}{(1-\alpha)\frac{Y_t(f)}{L_t^d(f)}}. \tag{39}$$

This leads to

$$\int_0^1 L_t^d(f)df = (1-\alpha)\frac{\sigma_t}{\zeta_t}\int_0^1 Y_t(f)df. \tag{40}$$

3.5 First-Order Condition for Intermediate Prices

The pricing decisions of firms are subject to Calvo-style contracts. In any period, an intermediate firm f can renew its price $P_t(f)$ with probability $1-\xi_p$. If a firm obtains the Calvo signal in period t, but not in any of the periods between t and $t+j$, then the firm resets its price according to $P_{t+s}(f) = P_t\pi^s$, for all s between 1 and j.

Below, we consider the pricing decision of a firm that gets to renew its price in period t. Its profit maximization problem can then be written as:

$$\max_{[P_t(f)]} E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t}\left[(1+\tau_p)\pi^j P_t(f)Y_{t+j}(f) - \Sigma_{t+j}Y_{t+j}(f)\right]. \tag{41}$$

From the profit maximization problem

$$\begin{aligned} \frac{\partial}{\partial P_t(f)} &= E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t}\Big[(1+\tau_p)\pi^j Y_{t+j}(f) + \\ &\quad +(1+\tau_p)\pi^j \frac{\partial Y_{t+j}(f)}{\partial P_t(f)} P_t(f) - \Sigma_{t+j}\frac{\partial Y_{t+j}(f)}{\partial P_t(f)}\Big] = 0 \end{aligned} \tag{42}$$

Rewrite the above in terms of the price elasticity of demand. From the aggregate demand function for product f we know that $\frac{\partial Y_t(f)}{\partial P_t(f)}\frac{P_t}{Y_t} = -\frac{1+\theta_p}{\theta_p}$. Similarly, notice also that $\frac{\partial Y_{t+j}(f)}{\partial P_t(f)}\frac{P_t}{Y_t} = -\frac{1+\theta_p}{\theta_p}$. We can then exploit the constant elasticity feature of demand to simplify the first-order condition.

$$E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t}\left[(1+\tau_p)\pi^j Y_{t+j}(f)\left(1+\frac{\frac{\partial Y_{t+j}(f)}{\partial P_t(f)}}{\frac{Y_{t+j}(f)}{P_t(f)}}\right)\right] = E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t}\Sigma_{t+j}\frac{Y_{t+j}(f)}{P_t(f)}\frac{\frac{\partial Y_{t+j}(f)}{\partial P_t(f)}}{\frac{Y_{t+j}(f)}{P_t(f)}} \tag{43}$$

Substituting for the elasticity value and multiplying by $P_t(f)$:

$$P_t(f)E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t}\left[(1+\tau_p)\pi^j Y_{t+j}(f)(-\frac{1}{\theta_p})\right] = E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t}\Sigma_{t+j}Y_{t+j}(f)\left(-\frac{1+\theta_p}{\theta_p}\right) \tag{44}$$

Dividing both sides by P_t, and collecting the θ_p terms (remember that $\sigma_t = \frac{\Sigma_t}{P_t}$)

$$\frac{P_t(f)}{P_t} E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t} \frac{(1+\tau_p)}{(1+\theta_p)} \pi^j Y_{t+j}(f) = E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t} \sigma_{t+j} \frac{P_{t+j}}{P_t} Y_{t+j}(f) \tag{45}$$

Let the relative contract price, $P_{r,t}$, be defined as $P_{r,t} = \frac{P_t(f)}{P_t}$. From the aggregate demand for the product of firm f we know that $Y_{t+j}(f) = Y_{t+j} \left(\frac{P_t(f)}{P_t} \frac{P_t \pi^j}{P_{t+j}} \right)^{-\frac{1+\theta_p}{\theta_p}}$. Combining this equation for $Y_{t+j}(f)$ with the one above

$$\frac{(1+\tau_p)}{(1+\theta_p)} P_{r,t} E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t} \pi^j Y_{t+j} \left(\frac{P_t(f)}{P_t} \frac{P_t \pi^j}{P_{t+j}} \right)^{-\frac{1+\theta_p}{\theta_p}} = \tag{46}$$
$$E_t \sum_{j=0}^{\infty} \xi_p^j \psi_{t+j,t} \sigma_{t+j} \frac{P_{t+j}}{P_t} Y_{t+j} \left(\frac{P_t(f)}{P_t} \frac{P_t \pi^j}{P_{t+j}} \right)^{-\frac{1+\theta_p}{\theta_p}}$$

The stochastic discount factor, $\psi_{t+j,t}$ is such that $E_t \psi_{t+j,t} = \beta^j \frac{P_t}{P_{t+j}} \frac{\Lambda_{t+j}}{\Lambda_t}$ (see the first-order condition for bond holding).

$$\frac{(1+\tau_p)}{(1+\theta_p)} P_{r,t} E_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \frac{P_t}{P_{t+j}} \frac{\Lambda_{t+j}}{\Lambda_t} \pi^j Y_{t+j} \left(\frac{P_t(f)}{P_t} \frac{P_t \pi^j}{P_{t+j}} \right)^{-\frac{1+\theta_p}{\theta_p}} = \tag{47}$$
$$E_t \sum_{j=0}^{\infty} \xi_p^j \beta^j \frac{P_t}{P_{t+j}} \frac{\Lambda_{t+j}}{\Lambda_t} \sigma_{t+j} \frac{P_{t+j}}{P_t} Y_{t+j} \left(\frac{P_t(f)}{P_t} \frac{P_t \pi^j}{P_{t+j}} \right)^{-\frac{1+\theta_p}{\theta_p}}$$

Collecting terms and rearranging

$$\frac{(1+\tau_p)}{(1+\theta_p)} P_{r,t} \sum_{j=0}^{\infty} (\xi_p \beta)^j \frac{\Lambda_{t+j}}{\Lambda_t} Y_{t+j} \frac{P_t \pi^j}{P_{t+j}} \left(\frac{P_t \pi^j}{P_{t+j}} \right)^{-\frac{1+\theta_p}{\theta_p}} = \tag{48}$$
$$\sum_{j=0}^{\infty} (\xi_p \beta)^j \frac{\Lambda_{t+j}}{\Lambda_t} \sigma_{t+j} Y_{t+j} \frac{P_t}{P_{t+j}} \frac{P_{t+j}}{P_t} \left(\frac{P_t \pi^j}{P_{t+j}} \right)^{-\frac{1+\theta_p}{\theta_p}}$$

Simplifying:

$$\frac{(1+\tau_p)}{(1+\theta_p)} P_{r,t} E_t \sum_{j=0}^{\infty} (\xi_p \beta)^j \frac{\Lambda_{t+j}}{\Lambda_t} Y_{t+j} \left(\frac{P_{t+j}}{P_t \pi^j} \right)^{\frac{1}{\theta_p}} = \tag{49}$$
$$E_t \sum_{j=0}^{\infty} (\xi_p \beta)^j \frac{\Lambda_{t+j}}{\Lambda_t} \sigma_{t+j} Y_{t+j} \left(\frac{P_{t+j}}{P_t \pi^j} \right)^{\frac{1+\theta_p}{\theta_p}}$$

Define $\pi_{t+j} = \frac{P_{t+j}}{P_{t+j-1}\pi}$. Loglinearizing:

$$\frac{1+\tau_p}{1+\theta_p} P_r y \left[\sum_{j=0}^{\infty} (\beta\xi_p)^j \left[\hat{\Lambda}_{c,t+j} + \hat{y}_{t+j} \right] + \frac{1}{\theta_p} \sum_{j=1}^{\infty} (\beta\xi_p)^j \sum_{k=1}^{j} \hat{\pi}_{t+k} \right] + \frac{1+\tau_p}{1+\theta_p} \frac{P_r y}{1-\beta\xi_p} \hat{P}_{r,t} =$$
$$\sigma y \sum_{j=0}^{\infty} (\beta\xi_p)^j \left[\hat{\Lambda}_{c,t+j} + \hat{\sigma}_{q,t+j} + \hat{y}_{t+j} \right] + \sigma y \left(\frac{1+\theta_p}{\theta_p} \right) \sum_{j=1}^{\infty} (\beta\xi_p)^j \sum_{k=1}^{j} \hat{\pi}_{t+k}.$$

Notice that I have dropped the conditional expectation operator. The log-linear terms whose time subscript is greater than t are understood to be in conditional expectation form, given information available at time t. Also notice that in steady state $\frac{1+\tau_p}{1+\theta_p} P_r = \sigma$. Collecting terms

$$\frac{\hat{P}_{r,t}}{1-\beta\xi_p} = \sum_{j=0}^{\infty} (\beta\xi_p)^j \hat{\sigma}_{t+j} + \sum_{j=1}^{\infty} (\beta\xi_p)^j \sum_{k=1}^{j} \hat{\pi}_{t+k} \tag{50}$$

Following the same reasoning as for the wage equation, one can show that the above equation implies

$$\hat{\pi}_t = \beta \hat{\pi}_{t+1} + \kappa_p \hat{\sigma}_t \tag{51}$$

where $\kappa_p = \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}$

4 The Government

The main role of the government is that of determining the reaction of the nominal interest rate i_t to the realization of different states of nature.

We are going to assume that the rule for interest rate setting takes the form:

$$i_t = \frac{\pi}{\beta} - 1 + \psi_\pi (\pi_t - \pi) + \psi_y (log(y_t) - log(y_{t-1})). \tag{52}$$

In a few classes we are going to study the differences between constrained ad-hoc rules such as the one above and the optimal unconstrained response of the government given a particular loss function.

Notice that households in this setting are Ricardian. The time profile of lump-sum taxes does not alter their decisions. Hence, we do not need to specify a specific form for the lump-sum tax rate reaction function.

5 Aggregation and Resource Constraints

From the first-order conditions for the cost minimization problem of intermediate firms 38 we had obtained that

$$\sigma_t = \frac{r_{k,t}}{\alpha \frac{Y_t(f)}{K_t(f)}} \tag{53}$$

rearranging

$$\alpha \frac{\sigma_t}{r_{k,t}} Y_{t,f} = K_t(f) \tag{54}$$

Aggregating over firms

$$\alpha \frac{\sigma_t}{r_{k,t}} \int_f Y_t(f) = \int_0^1 K_t(f) df \tag{55}$$

Using the intermediate product demand equation

$$\alpha \frac{\sigma_t}{r_{k,t}} Y_t \int_f \left(\frac{P_t(f)}{P_t} \right)^{-\frac{1+\theta_p}{\theta_p}} = K_t \tag{56}$$

Next, we shall see that the price terms drop out from a first-order approximation around a symmetric steady state in which all prices are 1, in other words: $\int_f \hat{y}_t(f) = \hat{y}_t$. Log linearizing:

$$-\frac{1+\theta_p}{\theta_p} \hat{P}_t = \widehat{\left[\int_f P_t(f)^{-\frac{1+\theta_p}{\theta_p}} \right]^{-\frac{\theta_p}{1+\theta_p}}} = -\frac{1+\theta_p}{\theta_p} \int_f \hat{P}_t(f) [P(f)]^{-\frac{\theta_p}{1+\theta_p}-1} = -\frac{1+\theta_p}{\theta_p} \int_f \hat{P}_t(f). \tag{57}$$

From this we can see that $P_t \approx \int_f P_t(f)$. Accordingly, the equation below

$$\alpha \frac{\sigma_t}{r_{k,t}} Y_t \approx K_t \tag{58}$$

is first-order equivalent to equation (56).

By the same logic, to a first-order approximation, we can also write the resource constraint as:

$$\int_f Y_t(f) \approx Y_t = C_t + I_t. \tag{59}$$

Similarly, in the labor market:

$$\int_h L_t(h) \approx L_t = \int_f L_t(f). \tag{60}$$