Handout 1

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1 Solving a Linear Approximation of the Real Business Cycle model with Fixed Labor Supply

This handout is meant as a guide to the steps involved in obtaining the first-order approximation to the solution of a dynamic stochastic general equilibrium model. As an example, it considers a simple model of real business cycles with fixed labor supply.

1.1 Model description

Households seek to maximize utility given by:

$$\sum_{t=0}^{\infty} E_t \beta^t log(c_t).$$

Households have access to a production technology given by:

$$y_t = e^{z_t} k_t^{\alpha},$$

where z_t is a shock process governed by:

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \tag{1}$$

where ϵ_{t+1} is normally and independently distributed with mean 0 and variance σ^2. The law of motion for capital is

$$k_{t+1} = (1-\delta)k_t + i_t.$$

Finally, the resource constraint for the economy implies that

$$c_t + i_t = y_t.$$

1.2 Necessary conditions for an equilibrium

To find the necessary conditions for an equilibrium setup the households maximization problem using the following Lagrangian:

$$\max_{c_t, k_{t+1}, i_t, \lambda_t, \gamma_t} \quad L = \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right. \tag{2}$$

$$+\beta^t \lambda_t \left[e^{z_t} k_t^{\alpha} - c_t - i_t \right] \tag{3}$$

$$\left. +\beta^t \gamma_t \left[k_{t+1} - (1-\delta) k_t - i_t \right] \right\} \tag{4}$$

N.B.: the way in which you write the lagrangian constraints affects the interpretation of the multiplier, but does not affect the final solution.

The first-order conditions of the Lagrangian with respect to the maximization objects above are given by:

$$\frac{\partial L}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \tag{5}$$

$$\frac{\partial L}{\partial k_{t+1}} = \beta E_t \left[\lambda_{t+1} \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} \right] + \gamma_t - \beta E_t \left[\gamma_{t+1} (1-\delta) \right] = 0 \tag{6}$$

$$\frac{\partial L}{\partial i_t} = -\lambda_t - \gamma_t = 0 \tag{7}$$

$$\frac{\partial L}{\partial \lambda_t} = e^{z_t} k_t^{\alpha} - c_t - i_t = 0 \tag{8}$$

$$\frac{\partial L}{\partial \gamma_t} = k_{t+1} - (1-\delta) k_t - i_t = 0 \tag{9}$$

The necessary conditions for an equilibrium of the model are given by all the first-order conditions above, plus the shock process in equation (1).

1.3 Some manual intervention

Before attempting to solve the model, realize that the conditions for an equilibrium listed above can be reduced to a smaller set of equations. This manual intervention is simple in the case of this model, but might be substantially more involved for other models. Later, we are going to learn how to deploy some numerical techniques to avoid it altogether.

Using equation (7) and equation (5) notice that

$$\lambda_t = \frac{1}{c_t}, \quad \gamma_t = -\frac{1}{c_t}. \tag{10}$$

Substituting λ_t and γ_t from equations (10) into equation (6) and collecting terms, we obtain:

$$\beta E_t \left[\frac{1}{c_{t+1}} \left(1 - \delta + \alpha e^{z_{t+1}} k_{t+1}^{(\alpha-1)}\right) \right] = \frac{1}{c_t}. \tag{11}$$

Solving the resource constraint in equation (8) for i_t and substituting in equation (9) one obtains

$$k_{t+1} = (1-\delta)k_t + e^{z_t} k_t^{\alpha} - c_t. \tag{12}$$

We have now expressed the necessary conditions for an equilibrium as three equations (11), (12), and (1) and three variables k,c, and z.

1.4 Model calibration and non-stochastic steady states

Before we can compute the steady state values of k, c, and z, we need to choose numerical values for the parameters in the model. Let $\delta = 0.025$, $\beta = 0.99$, $\alpha = 0.33$, and $\rho = 0.95$.

Let "*" denote steady state values. From equation (1), $z^* = 0$. Working on equation (11)

$$\beta \left[\frac{1}{c^*} (1 - \delta + \alpha k^{*\alpha-1}) \right] = \frac{1}{c^*}. \tag{13}$$

This implies that

$$k^* = \left[\frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta \right) \right]^{\frac{1}{\alpha-1}} \tag{14}$$

Finally, from equation(12), $k^* = (1-\delta)k^* + k^{*\alpha} - c^*$, which yields

$$c^* = k^{*\alpha} - \delta k^*. \tag{15}$$

Using the parameter choices above, the numerical steady states are: $z^* = 0$, $k^* \approx 28.3$, $c^* \approx 2.3$.

1.5 Linearizing the model

Let a "^" denote a variable's deviation from its non-stochastic steady state value, i.e., $\hat{c}_t = c_t - c^*$. Using the first-order Taylor series expansion around the non-stochastic steady state, one can express the necessary conditions for the model's equilibrium as:

$$-\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1})E_t\hat{c}_{t+1} + \frac{\beta}{c^*}\alpha(\alpha-1)k^{*(\alpha-2)}E_t\hat{k}_{t+1} + \frac{\beta}{c^*}\alpha k^{*\alpha-1}E_t\hat{z}_{t+1} = -\frac{1}{c^{*2}}\hat{c}_t \quad (16)$$

$$\hat{k}_{t+1} = -\hat{c}_t + (1-\delta+\alpha k^{*\alpha-1})\hat{k}_t + k^{*\alpha}\hat{z}_t \quad (17)$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \hat{\epsilon}_{t+1} \quad (18)$$

Next, deal with the conditional expectation operator. Start with $E_t c_{t+1}$. Consider the identity: $E_t c_{t+1} = c_{t+1} + (E_t c_{t+1} - c_{t+1})$. This is simply saying that current expectation of next period's consumption can be expressed as the realization of next period's consumption plus an expectational error. Trivially, this relationship will also hold in deviation from steady state: $E_t\hat{c}_{t+1} = \hat{c}_{t+1} + (E_t\hat{c}_{t+1} - \hat{c}_{t+1})$. Now, introduce a little more notation. Let

$$\hat{\omega}_{ct+1} = (E_t\hat{c}_{t+1} - \hat{c}_{t+1}), \quad (19)$$

$$\hat{\omega}_{kt+1} = (E_t\hat{k}_{t+1} - \hat{k}_{t+1}), \quad (20)$$

$$\hat{\omega}_{zt+1} = (E_t\hat{z}_{t+1} - \hat{z}_{t+1}). \quad (21)$$

Solve definitions (19) to (21) for $E_t\hat{c}_{t+1}$, $E_t\hat{k}_{t+1}$, and $E_t\hat{z}_{t+1}$, respectively and substitute into equation (16). Then, one obtains:

$$\begin{aligned}
&-\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1})[\hat{c}_{t+1} + \hat{\omega}_{ct+1}] \\
&+\frac{\beta}{c^*}\alpha(\alpha-1)k^{*(\alpha-2)}[\hat{k}_{t+1} + \omega_{\hat{kt+1}}] \\
&+\frac{\beta}{c^*}\alpha k^{*\alpha-1}[\hat{z}_{t+1} + \hat{\omega}_{zt+1}] = -\frac{1}{c^{*2}}\hat{c}_t
\end{aligned} \quad (22)$$

Rewrite equations (22), (17), (18) in matrix form:

$$\psi\hat{x}_{t+1} + J\hat{\omega}_{t+1} = \phi\hat{x}_t + \hat{e}_{t+1}, \quad (23)$$

where

$$\hat{x}_s = \begin{pmatrix} \hat{c}_s \\ \hat{k}_s \\ \hat{z}_s \end{pmatrix}, \quad \hat{\omega}_s = \begin{pmatrix} \hat{\omega}_{cs} \\ \hat{\omega}_{ks} \\ \hat{\omega}_{zs} \end{pmatrix}, \quad \hat{e}_s = \begin{pmatrix} 0 \\ 0 \\ \hat{\epsilon}_s \end{pmatrix}$$

and where

$$\psi = \begin{pmatrix} -\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1}) & \frac{\beta}{c^*}\alpha(\alpha-1)k^{*(\alpha-2)} & \frac{\beta}{c^*}\alpha k^{*\alpha-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} -\frac{1}{c^{*2}} & 0 & 0 \\ -1 & (1-\delta+\alpha k^{*\alpha-1}) & k^{*\alpha} \\ 0 & 0 & \rho \end{pmatrix},$$

where ψ_{11}, ψ_{12}, and $\psi_{1,3}$ denote the relevant entries of the matrix ψ. Equation (23) can be simplified a little more, so as to yield:

$$\psi\hat{x}_{t+1} = \phi\hat{x}_t + \hat{f}_{t+1} \quad with \;\; \hat{f}_{t+1} = \begin{pmatrix} -\psi_{row(1)}\hat{\omega}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}, \tag{24}$$

where $\psi_{row(1)}$ denotes the first row of the matrix ψ.

From here onwards, we need to proceed numerically.

1.6 Case 1: ψ is invertible

The simplifications we brought to the necessary conditions for an equilibrium have ensured that ψ is invertible.

Premultiplying equation (24) by ψ^{-1}, one can see that:

$$\hat{x}_{t+1} = \psi^{-1}\phi\hat{x}_t + \psi^{-1}\hat{f}_{t+1} \tag{25}$$

You might be tempted to stop here, but you'd not be quite done yet. Remember that $\hat{f}_{t+1}$ is a function of expectational errors. We need to solve for those. We'll get there in what might initially appear a rather circuitous route.

Let $A = \psi^{-1}\phi$. The next step is to find the eigenvalues and eigenvectors of A. Place the eigenvalues of A along the diagonal of the matrix D. Arrange the corresponding eigenvectors of A along the columns of the matrix V. Fortunately you don't have to do this by hand. Matlab provides the function *eig* to construct the matrices V and D. Premultiplying equation (25) by V^{-1}, thus

$$V^{-1}\hat{x}_{t+1} = V^{-1}A\hat{x}_t + V^{-1}\psi^{-1}\hat{f}_{t+1}$$

Remember that $AV = VD$. Thus, $V^{-1}AV = D$. But then, $V^{-1}A = DV^{-1}$. Accordingly:

$$V^{-1}\hat{x}_{t+1} = DV^{-1}\hat{x}_t + V^{-1}\psi^{-1}\hat{f}_{t+1} \tag{26}$$

Next, change variables, let $\hat{y}_s = V^{-1}\hat{x}_s$. For the new variables, we can rewrite the equation (26) above as:

$$\hat{y}_{t+1} = D\hat{y}_t + V^{-1}\psi^{-1}\hat{f}_{t+1}. \tag{27}$$

This last transformation puts us in a really good position. We have written the necessary conditions for an equilibrium in our model so that each condition involves only one variable (albeit, a linear combination of the original variables) and some linear combination of the expectational errors and the innovation to technology. Let the diagonal entries of D be denoted by d_i. Furthermore, let $\eta = V^{-1}\psi^{-1}$. Accordingly, each equation can be written as:

$$\hat{y}_{it+1} = d_i\hat{y}_{it} + \eta_{row(i)}\hat{f}_{t+1}. \tag{28}$$

Notice that if $|d_i| > 1$, then taking the conditional expectation at time t and iterating on (28) implies that $E_t\hat{y}_{is}$ might eventually explode, i.e. $\lim_{s\to\infty}|E_t\hat{y}_{is}| = \infty$ under some conditions. When $|d_i| > 1$, the d_i eigenvalue is said to be explosive.

To see the argument more clearly:

$$E_t\hat{y}_{it+1} = d_iE_t\hat{y}_{it} + E_t\eta_{row(i)}\hat{f}_{t+1}, \tag{29}$$

but by rational expectations the current expectation of future expectational errors is 0. Thus, $E_t \eta_{row(i)} \hat{f}_{t+1} = 0$ and

$$E_t \hat{y}_{it+1} = d_i \hat{y}_{it}. \tag{30}$$

Iterating forward

$$\frac{1}{d_i} E_{t+1} \hat{y}_{it+2} = \hat{y}_{it+1} \tag{31}$$

and substituting into (30), one can see that

$$E_t E_{t+1} \hat{y}_{it+2} = d_i^2 \hat{y}_{it}. \tag{32}$$

Using the law of iterated expectations:

$$E_t \hat{y}_{it+2} = d_i^2 \hat{y}_{it}. \tag{33}$$

Iterating forward some more, the power on the term d_i will keep growing, which is the basis for the claim of explosiveness above when $\hat{y}_{it}$ is nonzero.

Now, suppose we impose a non-explosiveness condition of the form $\lim_{s \to \infty} E_t y_i s = 0$. The interpretation of this condition is that far enough in the future, we should expect that all variables converge in expectation to their non-stochastic steady state values. If $|d_i| > 1$, this condition can only be satisfied if $\hat{y}_{it}$ is 0 at all times.

Given our parametric choices, we can confirm that, for the model we are considering, we indeed have one explosive eigenvalue (and 2 non-explosive eigenvalues), which ensures that there will be a unique rational expectation equilibrium in the neighborhood of the non-stochastic steady state (see Blanchard and Kahn). Then, we know that for some i, $\hat{y}_{it} = 0 \ \forall \ t$. This can buy us two interesting simplifications.

Simplification # 1

How about the expectational errors? Given that $y_{it} = 0 \ \forall \ t$, where i is such that $|d_i| > 1$, from equation (28) we have that

$$\eta_{row(i)} \hat{f}_{t+1} = 0.$$

Remembering the definition of $\hat{f}$, the above implies

$$\eta_{row(i)} \begin{pmatrix} -\psi_{row(1)}\hat{\omega}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix} = 0, \tag{34}$$

which can be used to solve for $\psi_1\hat{\omega}_{t+1}$. Thus,

$$\psi_1\hat{\omega}_{t+1} = \frac{\eta_{i,3}}{\eta_{i,1}}\hat{\epsilon}_{t+1}. \tag{35}$$

The above equation has some interesting economic interpretation. The expectational errors are linearly related to the innovation to the productivity shock process $\hat{\epsilon}_{t+1}$.

At this point we could declare victory. For any set of predetermined conditions $\hat{x}_t = (\hat{c}_t, \hat{k}_t, \hat{z}_t)'$, and innovation $\hat{\epsilon}_t$, we can use V^{-1} to map those conditions into a vector $\hat{y}_t$. Using the restriction in equation (35), we can construct $\hat{f}_{t+1}$. Then, using equation (27) we can obtain $\hat{y}_{t+1}$, which can be transformed back into $\hat{x}_{t+1}$ using V.

Simplification # 2

While we could have stopped at simplification #1, a little more algebra will yield even more rewards.

Since $\hat{y}_{is} = (V^{-1})_{row(i)}\hat{x}_s$, imposing the non-explosiveness condition, we have that:

$$y_{it} = \sum_j (V^{-1})_{ij} = (V^{-1})_{i1}c_t + (V^{-1})_{i2}k_t + (V^{-1})_{i3}z_t = 0 \tag{36}$$

We can use this relationship to solve for one of the variables in terms of the other two. Pick $\hat{c}_t$. Thus,

$$\hat{c}_t = \begin{pmatrix} c_k & c_z \end{pmatrix} \begin{pmatrix} k_t \\ z_t \end{pmatrix}, \tag{37}$$

where $c_k = -\frac{(V^{-1})_{i2}}{(V^{-1})_{i1}}$ and $c_z = -\frac{(V^{-1})_{i3}}{(V^{-1})_{i1}}$. In general, for each explosive eigenvalue, we can solve one variable out of the system.

Finally, we can substitute the restrictions we have found for $\hat{c}_t$ and $\psi_1\hat{\omega}_{t+1}$ into the the original system of equations in matrix form. Focus on the second and third equation in (25).

They can be rewritten as:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} = \begin{pmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{z}_t \end{pmatrix} + \begin{pmatrix} \psi^{-1}_{row(2)} \\ \psi^{-1}_{row(3)} \end{pmatrix} \begin{pmatrix} -\frac{\eta_{i,3}}{\eta_{i,1}} \hat{\epsilon}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}.$$

Substituting equation (37) and (35), into the above:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} = \left(\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} \begin{pmatrix} c_k & c_z \end{pmatrix} + \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \right) \begin{pmatrix} \hat{k}_t \\ \hat{z}_t \end{pmatrix} + \begin{pmatrix} \psi^{-1}_{row(2)} \\ \psi^{-1}_{row(3)} \end{pmatrix} \begin{pmatrix} -\frac{\eta_{i,3}}{\eta_{i,1}} \hat{\epsilon}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}.$$

Remember that the last two rows of ψ had ones along the diagonal and zeros everywhere else. That's going to carry through to its inverse, ψ^{-1}. So, $(\psi^{-1})_{row(2)} = \{0, 1, 0\}$, and $(\psi^{-1})_{row(3)} = \{0, 0, 1\}$, which implies:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} = \left(\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} \begin{pmatrix} c_k & c_z \end{pmatrix} + \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \right) \begin{pmatrix} \hat{k}_t \\ \hat{z}_t \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}.$$