# Handout 3

February 3, 2010

## 1 Introducing a Stochastic Trend in the RBC Model

Consider the same basic model as in Handout 1, but let's make a couple of changes pertaining the shock process. Instead of an AR(1) process in the level of technology, this time consider a set up in which the growth rate of technology is not necessarily zero in steady state and is itself governed by a stationary AR(1) process.

### 1.1 Model description

Households seek to maximize utility given by:

$$\sum_{t=0}^{\infty} E_t \beta^t log(c_t).$$

Households have access to a production technology given by:

$$y_t = z_t^{1-\alpha} k_t^{\alpha},$$

where $z_t$ evolves according to:

$$z_{t+1} = \mu_{zt+1} z_t. \tag{1}$$

In turn, $\mu_{zt+1}$ is governed by:

$$\mu_{zt+1} - \mu_z^* = \rho_{\mu_z}(\mu_{zt+1} - \mu_z^*) + \epsilon_{\mu_z t}, \tag{2}$$

and where $\epsilon_{\mu_z t}$ is normally and independently distributed. The law of motion for capital is

$$k_{t+1} = (1-\delta)k_t + i_t.$$

Finally, the resource constraint for the economy implies that

$$c_t + i_t = y_t.$$

## 1.2 Necessary conditions for an equilibrium

To find the necessary conditions for an equilibrium setup the households maximization problem using the following Lagrangian:

$$\max_{c_t,k_{t+1},i_t,\lambda_t,\gamma_t} \quad L = \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right. \tag{3}$$

$$+\beta^t \lambda_t \left[ z_t^{1-\alpha} k_t^{\alpha} - c_t - i_t \right] \tag{4}$$

$$\left. +\beta^t \gamma_t \left[ k_{t+1} - (1-\delta)k_t - i_t \right] \right\} \tag{5}$$

The first-order conditions of the Lagrangian with respect to the maximization objects above are given by:

$$\frac{\partial L}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \tag{6}$$

$$\frac{\partial L}{\partial k_{t+1}} = \beta E_t \left[ \lambda_{t+1} \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right] + \gamma_t - \beta E_t \left[ \gamma_{t+1}(1-\delta) \right] = 0 \tag{7}$$

$$\frac{\partial L}{\partial i_t} = -\lambda_t - \gamma_t = 0 \tag{8}$$

$$\frac{\partial L}{\partial \lambda_t} = z_t^{1-\alpha} k_t^{\alpha} - c_t - i_t = 0 \tag{9}$$

$$\frac{\partial L}{\partial \gamma_t} = k_{t+1} - (1-\delta)k_t - i_t = 0 \tag{10}$$

## 1.3 Normalizations

Because $z_t$ is expected to grow through time, the model does not have a non-stochastic steady state for the original variables we used in setting up the Lagrangian problem. However, we can easily find a transformation of those variables that will have a steady state. Consider then the following changes in variables:

$$\begin{aligned}\frac{i_t}{z_t} &= \tilde{i}_t \\ \frac{c_t}{z_t} &= \tilde{c}_t \\ \frac{y_t}{z_t} &= \tilde{y}_t \\ \frac{k_{t+1}}{z_t} &= \tilde{k}_{t+1}.\end{aligned} \tag{11}$$

Using equation (8) and equation (6) notice that

$$\lambda_t = \frac{1}{c_t}, \quad \gamma_t = -\frac{1}{c_t}. \tag{12}$$

Substituting $\lambda_t$ and $\gamma_t$ from equations (12) into equation (7) and collecting terms, we obtain:

$$\beta E_t \left[ \frac{1}{c_{t+1}} \left( 1 - \delta + \alpha z_{t+1}^{1-\alpha} k_{t+1}^{(\alpha-1)} \right) \right] = \frac{1}{c_t}. \tag{13}$$

Multiplying through by $E_t z_{t+1}$

$$\beta E_t \left[ \frac{z_{t+1}}{c_{t+1}} \left( 1 - \delta + \alpha \left( \frac{z_{t+1}}{z_t} \right)^{1-\alpha} \left( \frac{k_{t+1}}{z_t} \right)^{(\alpha-1)} \right) \right] = \frac{z_t}{c_t} E_t \frac{z_{t+1}}{z_t}. \tag{14}$$

Using the changes in variables listed above

$$\beta E_t \left[ \frac{1}{\tilde{c}_{t+1}} \left( 1 - \delta + \alpha \left( \mu_{zt} \right)^{1-\alpha} \left( \tilde{k}_{t+1} \right)^{(\alpha-1)} \right) \right] = \frac{1}{\tilde{c}_t} E_t \mu_{zt+1}. \tag{15}$$

Solving the resource constraint in equation (9) for $i_t$ and substituting in equation (10) one obtains

$$k_{t+1} = (1-\delta) k_t + z_t^{1-\alpha} k_t^{\alpha} - c_t. \tag{16}$$

Dividing through by $z_t$, one can see that:

$$\frac{k_{t+1}}{z_t} = (1-\delta) \frac{k_t}{z_{t-1}} \frac{z_{t-1}}{z_t} + \left( \frac{k_t}{z_t} \right)^{\alpha} - \frac{c_t}{z_t}. \tag{17}$$

The equation above can be re-written as:

$$\frac{k_{t+1}}{z_t} = (1-\delta) \frac{k_t}{z_{t-1}} \frac{z_{t-1}}{z_t} + \left( \frac{k_t}{z_{t-1}} \right)^{\alpha} \left( \frac{z_{t-1}}{z_t} \right)^{\alpha} - \frac{c_t}{z_t} \tag{18}$$

and using the changes in variables suggested above

$$\tilde{k}_{t+1} = (1-\delta) \tilde{k}_t \frac{1}{\mu_{zt}} + \left( \tilde{k}_t \right)^{\alpha} \left( \frac{1}{\mu_{zt}} \right)^{\alpha} - \tilde{c}_t \tag{19}$$

Equations (2), (15), and (19) summarize the conditions for an equilibrium using stationary variables only. The solution of the model can then proceed as outlined in Handout 1.