Handout 4

February 3, 2010

1 Introducing a trend in the price of investment

Notice that the model described above implies that the price of investment is constant relative to the price of output. The paper by Greenwood, Herkowitz, and Krusell (GHK) listed in the syllabus noticed that U.S. data show a downward trend in the price of equipment and software and suggest a modeling shortcut to encompass this distinct trend while maintaining much of the model setup described above unchanged. While GHK consider a more elaborate model with multiple stocks of capital, the main modeling shortcut can be understood within the context of our simpler model with only one stock of capital. In the model below, the extension we are borrowing from GHK is that a given physical unit of investment might be more or less productive in generating installed capital used in production.

Under special conditions outlined by GHK, the model below generates the same aggregate variable as those of a two-sector model with distinct technologies for the production of investment and consumption goods. The beauty of the GHK shortcut is that we don't have to carry around the extra apparatus of a more general two-sector model.

1.1 Model description

Households seek to maximize utility given by:

$$\sum_{t=0}^{\infty} E_t \beta^t log(c_t).$$

Households have access to a production technology given by:

$$y_t = z_t^{1-\alpha} k_t^{\alpha},$$

where z_t evolves according to:

$$z_{t+1} = \mu_{zt+1} z_t. \tag{1}$$

In turn, μ_{zt+1} is governed by:

$$\mu_{zt+1} - \mu_z^* = \rho_{\mu_z}(\mu_{zt+1} - \mu_z^*) + \epsilon_{\mu_z t}, \tag{2}$$

and where $\epsilon_{\mu_z t}$ is normally and independently distributed. The law of motion for capital is

$$k_{t+1} = (1-\delta)k_t + q_t^{\frac{1-\alpha}{\alpha}} i_t$$

$$q_{t+1} = \mu_{qt+1} q_t. \tag{3}$$

In turn, μ_{qt+1} is governed by:

$$\mu_{qt+1} - \mu_q^* = \rho_{\mu_q}(\mu_{qt+1} - \mu_q^*) + \epsilon_{\mu_q t}, \tag{4}$$

and where $\epsilon_{\mu_q t}$ is normally and independently distributed.

Finally, the resource constraint for the economy implies that

$$c_t + i_t = y_t.$$

1.2 Equilibrium conditions

Proceeding to set up the Lagrangian problem much as described above, the first-order conditions for the maximization problem are given by:

$$\frac{\partial L}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \tag{5}$$

$$\frac{\partial L}{\partial k_{t+1}} = \beta E_t \left[\lambda_{t+1} \alpha z_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}\right] + \gamma_t - \beta E_t \left[\gamma_{t+1}(1-\delta)\right] = 0 \tag{6}$$

$$\frac{\partial L}{\partial i_t} = -\lambda_t - q_t^{\frac{1-\alpha}{\alpha}} \gamma_t = 0 \tag{7}$$

$$\frac{\partial L}{\partial \lambda_t} = z_t^{1-\alpha} k_t^{\alpha} - c_t - i_t = 0 \tag{8}$$

$$\frac{\partial L}{\partial \gamma_t} = k_{t+1} - (1-\delta)k_t - q_t^{\frac{1-\alpha}{\alpha}} i_t = 0 \tag{9}$$

1.3 Representing the system in terms of stationary variables

Consider then the following changes in variables:

$$\begin{aligned} \frac{i_t}{z_t q_t} &= \tilde{i}_t \\ \frac{c_t}{z_t q_t} &= \tilde{c}_t \\ \frac{y_t}{z_t q_t} &= \tilde{y}_t \\ \frac{k_{t+1}}{z_t q_t^{\frac{1}{\alpha}}} &= \tilde{k}_{t+1} \end{aligned} \tag{10}$$

Substituting 5 and 7 into equation 6 now yields:

$$\beta E_t \left[\frac{1}{c_{t+1}} \left((1-\delta) q_{t+1}^{\frac{\alpha-1}{\alpha}} + \alpha z_{t+1}^{1-\alpha} k_{t+1}^{(\alpha-1)} \right) \right] = \frac{1}{c_t} q_t^{\frac{\alpha-1}{\alpha}}.$$

Collecting $q_t^{\alpha-1\alpha}$

$$\beta E_t \left[\frac{q_{t+1}^{\frac{\alpha-1}{\alpha}}}{c_{t+1}} \left((1-\delta) + \alpha z_{t+1}^{1-\alpha} \left(\frac{k_{t+1}}{q_t^{\frac{1}{\alpha}}} \right)^{(\alpha-1)} \left(\frac{q_t}{q_{t+1}} \right)^{\frac{\alpha-1}{\alpha}} \right) \right] = \frac{1}{c_t} q_t^{\frac{\alpha-1}{\alpha}}.$$

Multiplying through by $q_t^{\frac{1}{\alpha}} E_t z_{t+1}$

$$\beta E_t \left[\frac{z_{t+1} q_{t+1}^{\frac{\alpha-1}{\alpha}}}{c_{t+1}} \frac{q_{t+1}^{\frac{1}{\alpha}}}{q_{t+1}^{\frac{1}{\alpha}}} q_t^{\frac{1}{\alpha}} \left((1-\delta) + \alpha \left(\frac{z_{t+1}}{z_t} \right)^{1-\alpha} \left(\frac{k_{t+1}}{z_t q_t^{\frac{1}{\alpha}}} \right)^{(\alpha-1)} \left(\frac{q_t}{q_{t+1}} \right)^{\frac{\alpha-1}{\alpha}} \right) \right] = \frac{z_t q_t}{c_t} E_t \frac{z_{t+1}}{z_t}.$$

Using the changes in variables described above:

$$\beta E_t \left[\frac{1}{\tilde{c}_{t+1}} \left(\frac{1}{\mu_{qt+1}} \right)^{\frac{1}{\alpha}} \left((1-\delta) + \alpha \mu_{zt+1}^{1-\alpha} \tilde{k}_{t+1}^{\alpha-1} \left(\frac{1}{\mu_{qt+1}} \right)^{\frac{\alpha-1}{\alpha}} \right) \right] = \frac{1}{\tilde{c}_t} E_t \mu_{zt+1}. \tag{11}$$

From (8)

$$z_t^{1-\alpha} \left(\frac{k_t}{q_t^{\frac{1}{\alpha}}} q_t^{\frac{1}{\alpha}} \right)^{\alpha} - c_t - i_t = 0.$$

Dividing through by $z_t q_t$

$$z_t^{-\alpha} z_{t-1}^{\alpha} \left(\frac{k_t}{z_{t-1} q_t^{\frac{1}{\alpha}}} \frac{q_{t-1}^{\frac{1}{\alpha}}}{q_{t-1}^{\frac{1}{\alpha}}} \right)^{\alpha} - \frac{c_t}{z_t q_t} - \frac{i_t}{z_t q_t} = 0.$$

Allowing for the changes in variables defined above:

$$\left(\frac{1}{\mu_{zt}} \right)^{\alpha} \frac{1}{\mu_{qt}} \left(\tilde{k}_t \right)^{\alpha} - \tilde{c}_t - \tilde{i}_t = 0. \tag{12}$$

Finally, dividing equation (9) by $z_t q_t^{\frac{1}{\alpha}}$

$$\frac{k_{t+1}}{z_t q_t^{\frac{1}{\alpha}}} - (1-\delta)\frac{k_t}{z_t q_t^{\frac{1}{\alpha}}} - q_t^{\frac{1-\alpha}{\alpha}} q_t^{-\frac{1}{\alpha}} \frac{i_t}{z_t} = 0$$

With a few more manipulations, one can see that

$$\frac{k_{t+1}}{z_t q_t^{\frac{1}{\alpha}}} - (1-\delta)\frac{k_t}{z_t q_t^{\frac{1}{\alpha}}} \frac{z_{t-1} q_{t-1}^{\frac{1}{\alpha}}}{z_{t-1} q_{t-1}^{\frac{1}{\alpha}}} - \frac{i_t}{z_t q_t} = 0,$$

and using the changes in variables defined above yields:

$$\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t \left(\frac{1}{\mu_{qt}}\right)^{\frac{1}{\alpha}} \frac{1}{\mu_{zt}} - \tilde{i}_t = 0 \tag{13}$$

1.4 Model calibration and non-stochastic steady states

Before we can compute the steady state values of $\tilde{k}$, $\tilde{c}$, $\tilde{i}$, and $\tilde{y}$, we need to choose numerical values for the parameters in the model. Let $\delta = 0.025$, $\beta = 0.99$, $\alpha = 0.33$, $\rho_{\mu_z} = \rho_{\mu_q} = 0.5$, $\mu_z^* = 1.005$, and $\mu_q^* = 1.005$. Let "*" denote steady state values.

Using equation 13

$$\tilde{i}^* = \tilde{k}^* \left(1 - \frac{1-\delta}{\mu_z^* \mu_q^{*\frac{1}{\alpha}}}\right) \tag{14}$$

Using equation 11, multiplying through by $\tilde{c}^*$, one obtains:

$$\beta \left(\frac{1}{\mu_q^*}\right)^{\frac{1}{\alpha}} \left((1-\delta) + \alpha \mu_z^{*1-\alpha} \tilde{k}^{*\alpha-1} \left(\frac{1}{\mu_q^*}\right)^{\frac{\alpha-1}{\alpha}}\right) = \mu_z^*.$$

Solving the equation above for $\tilde{k}^*$

$$\tilde{k}^* = \left(\frac{\frac{1}{\beta}\mu_q^{*\frac{1}{\alpha}} \mu_z^* - 1 + \delta}{\alpha \mu_z^{*1-\alpha} \left(\frac{1}{\mu_q^*}\right)^{\frac{\alpha-1}{\alpha}}}\right)^{\frac{1}{\alpha-1}} \tag{15}$$

Then from the production function:

$$\tilde{y}^* = \frac{1}{\mu_q^*}\left(\frac{\tilde{k}^*}{\mu_z^*}\right)^{\alpha} \tag{16}$$

and using the resource constraint

$$\tilde{c}^* = \tilde{y}^* - \tilde{i}^*. \tag{17}$$