Handout 2

1 A General Method for the Linear Approximation to the Solution of DSGE Models

The algebraic manipulations we performed onto the system of necessary conditions for an equilibrium ensured that the left-hand side matrix ψ would be invertible. In fact, if we had not substituted all the identities out of the system, ψ would not have been invertible. Ensuring the invertibility of ψ is too laborious a task in all but the simplest models we are interested in solving. Fortunately, we can deploy some more matrix algebra to help us.

To fix ideas, let's work with a very simple example of a setup in which ψ is not going to be invertible. Consider again the basic RBC model described earlier, but this time, rather than using the resource constraint $y_t = c_t + i_t$ to substitute out i_t from the capital accumulation equation $k_{t+1} = (1-\delta)k_t + i_t$, ignore this simplification. Upon linearization, the necessary conditions for an equilibrium in our model can then be written as:

$$-\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1})E_t\hat{c}_{t+1} + \frac{\beta}{c^*}\alpha(\alpha-1)k^{*(\alpha-2)}E_t\hat{k}_{t+1} + \frac{\beta}{c^*}\alpha k^{*\alpha-1}E_t\hat{z}_{t+1} = -\frac{1}{c^{*2}}\hat{c}_t \quad (1)$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t \quad (2)$$

$$0 = \hat{c}_t - \alpha k^{*\alpha-1}\hat{k}_t + \hat{i}_t - k^{*\alpha}\hat{z}_t \quad (3)$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \hat{\epsilon}_{t+1} \quad (4)$$

Proceeding analogously to our earlier case, rewrite equations (1) to (4) in matrix form. Howver, departing slightly from the earlier method, this time, if we are particular about the order in which we arrange equations and variables, it will have a big payoff later. The reasons, will become apparent in a little while.

We want to order variables and equations according to these rules:

1. Predetermined variables (e.g., values for shocks, capital stocks) come first, non predetermined variables (e.g., consumption) second.

2. Inter-temporal equations (e.g. law of motion for capital, shock processes) are placed above, intratemporal equations below.

3. The inter-temporal equations that have expectational errors, such as the Euler equation for consumption, are placed last among the set of intertemporal equations.

One ordering of the necessary conditions for an equilibrium for the model under study that satisfies the rules above is: (4), (2), (1), (3). One ordering for the variables in the system that satisfies the rules above is $\hat{z}_s, \hat{k}_s, \hat{c}_s, \hat{i}_s$.

Adopting the ordering above for equations and variables, we can write the system of necessary conditions for an equilibrium as:

$$\psi \hat{x}_{t+1} = \phi \hat{x}_t + \hat{f}_{t+1},$$

where

$$\hat{x}_s = \begin{pmatrix} \hat{z}_s \\ \hat{k}_s \\ \hat{c}_s \\ \hat{i}_s \end{pmatrix}, \quad \hat{f}_s = \begin{pmatrix} \hat{\epsilon}_s \\ 0 \\ -\psi_1 \hat{\omega}_s \\ 0 \end{pmatrix} \quad \hat{\omega}_s = \begin{pmatrix} \hat{\omega}_{zs} \\ \hat{\omega}_{ks} \\ \hat{\omega}_{cs} \\ \hat{\omega}_{is} \end{pmatrix},$$

$$\psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\beta}{c^*}\alpha k^{*\alpha-1} & \frac{\beta}{c^*}\alpha(\alpha-1)k^{*(\alpha-2)} & -\frac{\beta}{c^{*2}}(1-\delta+\alpha k^{*\alpha-1}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\phi = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 1 \\ 0 & 0 & -\frac{1}{c^{*2}} & 0 \\ -k^{*\alpha} & -\alpha k*^{\alpha-1} & 1 & 1 \end{pmatrix}.$$

Inspecting ψ, the last row of zeros (or the last column) make it singular.

Fortunately, we can still proceed much in the same way as when ψ is invertible, but we just need to be a little more patient with the algebra. For starters, use the generalized complex Schur decomposition on the matrices ϕ and ψ. If you have never used this decomposition before, don't panic, there is nothing too transcendental about it.

It is still a cause of great happiness to me that Matlab can yield the generalized Schur decomposition with the command *qz*. This decomposition can be applied to any pair of square matrices ψ and ϕ. It yields matrices $Q, Z, \psi_\triangle, \phi_\triangle$ that have some interesting properties:

$$\begin{aligned} Q\psi Z = \phi_\triangle, &\qquad Q^H \psi_\triangle Z^H, \\ Q\phi Z = \psi_\triangle, &\qquad Q^H \phi_\triangle Z^H, \\ Q^H Q = I, &\qquad Z^H Z = I, \end{aligned}$$

where, "^{H}" denotes the transpose of the complex conjugate, and I the identity matrix. One of the beautiful properties of this decomposition is that $\psi_\triangle$ and $\phi_\triangle$ are (upper) triangular matrices. Furthermore, the generalized eigenvalues of $\psi_\triangle$ and $\phi_\triangle$ are the same as those of the original matrices ψ and ϕ. In fact, the generalized eigenvalues can simply be found by dividing one by one the diagonal entries of $\psi_\triangle$ by the diagonal entries of $\phi_\triangle$.

The Schur decomposition is not unique. We want to reorder it so that the stable generalized eigenvalues are associated with the upper rows of $\psi_\triangle$ and $\phi_\triangle$, and the unstable generalized eigenvalues with the lower rows. This can be done with the Matlab command *ordqz*.

Let's go back to our system of conditions for an equilibrium in matrix form $\psi \hat{x}_{t+1} = \phi \hat{x}_t + \hat{f}_{t+1}$. We can now rewrite it as:

$$Q^H \psi_\triangle Z^H \hat{x}_{t+1} = Q^H \phi_\triangle Z^H \hat{x}_t + \hat{f}_{t+1}. \tag{5}$$

Premultiplying through by Q,

$$\psi_{\triangle} Z^H \hat{x}_{t+1} = \phi_{\triangle} Z^H \hat{x}_t + Q\hat{f}_{t+1}. \tag{6}$$

As you might predict, this invites a natural change in variables. Let $\hat{y}_s = Z^H \hat{x}_s$. Then,

$$\psi_{\triangle} \hat{y}_{t+1} = \phi_{\triangle} \hat{y}_t + Q\hat{f}_{t+1}. \tag{7}$$

Next let's partition $\hat{y}_s$ into two parts $\hat{y}_s = \begin{pmatrix} \hat{y}_{Ss} \\ \hat{y}_{Us} \end{pmatrix}$. The partition $\hat{y}_{Ss}$ has n_S rows, as many as the number of stable generalized eigenvalues of ψ and ϕ. The partition $\hat{y}_{Us}$ has n_U rows, as many as the number of unstable generalized eigenvalues. Applying the same partitioning to equation (7), one obtains:

$$\begin{pmatrix} \psi_{\triangle 11} & \psi_{\triangle 12} \\ \psi_{\triangle 21} & \psi_{\triangle 22} \end{pmatrix} \begin{pmatrix} \hat{y}_{St+1} \\ \hat{y}_{Ut+1} \end{pmatrix} = \begin{pmatrix} \phi_{\triangle 11} & \phi_{\triangle 12} \\ \phi_{\triangle 21} & \phi_{\triangle 22} \end{pmatrix} \begin{pmatrix} \hat{y}_{St} \\ \hat{y}_{Ut} \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ \hat{f}_{Ut+1} \end{pmatrix} \tag{8}$$

Thus, for example, $\psi_{\triangle 11}$ has dimensions $n_S \times n_S$.

Since all the entries of $\psi_{\triangle 21}$ and of $\phi_{\triangle 21}$ are zeros, the non-explosiveness condition implies that $\hat{y}_{Us} = 0$ for all s. To derive this result, notice that $\phi_{\triangle 22} V = \psi_{\triangle 22} V D$, where V collects the generalized eigenvectors of $\phi_{\triangle 22}$ and $\psi_{\triangle 22}$, and D is a diagonal matrix whose non-zero entries are the generalized eigenvalues. By construction, the generalized eigenvalues of $\psi_{\triangle 22}$ and $\phi_{\triangle 22}$ are unstable. So, we can construct a similar argument to the one we used in the previous section, in the case of ψ invertible, to show that the non-explosiveness condition does imply $\hat{y}_{Us} = 0$ for all s.

Note that the change in variables we used above also implies that $\hat{x}_s = Z\hat{y}_s$. Partititioning the system leads to:

$$\begin{pmatrix} \hat{x}_{St} \\ \hat{x}_{Ut} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \hat{y}_{St} \\ \hat{y}_{Ut} \end{pmatrix}$$

But, since $\hat{y}_{Us} = 0 \; \forall s$, we have that $\hat{x}_{St} = Z_{11}\hat{y}_{St}$, and also that $\hat{x}_{Ut} = Z_{21}\hat{y}_{St}$. Combining the previous two results and bringing the equation forward by one period:

$$\hat{x}_{Ut+1} = Z_{21} Z_{11}^{-1} \hat{x}_{St+1} \tag{9}$$

Notice that, by construction, if the Blanchard-Kahn conditions are satisfied, $\hat{x}_{Ut}$ only holds jump variables, and $\hat{x}_{St}$ only holds non-jump predetermined variables. So, equation (9) gives us a way do deduce all the jump variables from the predetermined variables, without any additional recourse to the innovations to the shock processes other than their effects already embedded in $\hat{x}_{St+1}$.

Finally, we need to solve for $\hat{x}_{St+1}$. From equation (8), using the result from the non-explosiveness condition that $\hat{y}_{Us} = 0 \; \forall \; s$, we have that

$$\psi_{\triangle 11}\hat{y}_{St+1} = \phi_{\triangle 11}\hat{y}_{St} + \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ \hat{f}_{Ut+1} \end{pmatrix} \tag{10}$$

But the non-explosiveness condition also implies

$$\begin{pmatrix} Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ \hat{f}_{Ut+1} \end{pmatrix} = 0,$$

from which one can see that

$$\hat{f}_{Ut+1} = -Q_{22}^{-1}Q_{21}\hat{f}_{St+1}. \tag{11}$$

Notice that, by construction, $\hat{f}_{St+1}$ will hold innovations to shocks and no expectational errors. Conversely, $\hat{f}_{Ut+1}$ will hold expectational errors, and no innovations. Accordingly, equation (11) gives us a way to retrieve expectational errors from fundamental innovations. The Blanchard-Kahn conditions, given our ordering rules, will ensure invertibility of Q_{22}.

Substituting equation (11) into equation (10), and remembering that $\hat{y}_{St} = Z_{11}^{-1}\hat{x}_{St}$, one obtains

$$\psi_{\triangle 11}(Z_{11})^{-1}\hat{x}_{St+1} = \phi_{\triangle 11}(Z_{11})^{-1}\hat{x}_{St} + \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix} \begin{pmatrix} \hat{f}_{St+1} \\ -Q_{22}^{-1}Q_{21}\hat{f}_{St+1} \end{pmatrix}.$$

Premultiplying by $Z_{11}\psi_{\triangle 11}^{-1}$ and collecting terms, then we can rewrite the equation above as

$$\hat{x}_{St+1} = Z_{11}\psi_{\triangle 11}^{-1}\phi_{\triangle 11}Z_{11}^{-1}\hat{x}_{St} + Z_{11}^{-1}\psi_{\triangle 11}\left(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}\right)\hat{f}_{St+1}. \tag{12}$$

Equations (9) and (12) give us a way to solve for all variables of interest. In fact, with just one more set of painless substitutions, they can be rewritten in a more visually appealing way:

$$\hat{x}_{St+1} = A\hat{x}_{St} + B\hat{f}_{St+1} \tag{13}$$

$$\hat{x}_{Ut+1} = C\hat{x}_{St+1} \tag{14}$$

$$A = Z_{11}\psi_{\triangle 11}^{-1}\phi_{\triangle 11}(Z_{11})^{-1} \tag{15}$$

$$B = Z_{11}\psi_{\triangle 11}^{-1}\left(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}\right) \tag{16}$$

$$C = Z_{21}(Z_{11})^{-1}, \tag{17}$$

which completes the "A,B,C" of model solving.