Solution to Problem Set 3

Question 1

In Handout 6, we saw that the conditional likelihood function takes the form:

$$\log(L) = -\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \sum_{t-2}^{T}\frac{(y_t - \rho y_{t-1})^2}{2\sigma^2}$$

Consider the first-order conditions for the maximization of the log-likelihood function:

$$\frac{\partial}{\partial\rho} = \sum\frac{(y_t - \rho y_{t-1})y_{t-1}}{2\sigma^2} = 0, \tag{1}$$

$$\frac{\partial}{\partial\sigma} = -\frac{T-1}{2}\frac{1}{\sigma^2}2\sigma + \sum\frac{(y_t - \rho y_{t-1})^2}{\sigma^3} = 0 \tag{2}$$

Solving the FOCs above for ρ and σ yields their conditional maximum likelihood estimates:

$$\hat{\rho}^{ML} = \frac{\sum y_t y_{t-1}}{\sum y_{t-1}^2} \tag{3}$$

$$\hat{\sigma}^{2ML} = \frac{\sum(y_t - \hat{\rho} y_{t-1})^2}{T-1} \tag{4}$$

Next consider the OLS minimization problem:

$$\min_{\rho}\sum_{t=2}^{T}\sum(y_t - \rho y_{t-1})^2.$$

The first-order condition for this problem yields:

$$\hat{\rho}^{OLS} = \frac{\sum y_t y_{t-1}}{\sum y_{t-1}^2}.$$

which coincides with the ML estimator derived above.

Next, consider the variance of the ML estimator for ρ and σ. We know that ML is efficient and will reach the Cramer-Rao lower bound. Therefore, we can use the inverse of the information matrix (minus the Hessian evaluated at the maximum of the likeihood) to form the asymptotic estimates for the variances of interest.

First, to form the Hessian, compute all the second derivatives of the log-likelihood.

$$\frac{\partial^2}{\partial\rho^2} = -\frac{\sum y_{t-1}^2}{\sigma^2}, \tag{5}$$

$$\frac{\partial^2}{\partial\rho\partial\sigma} = -\frac{\sum(y_t - \rho y_{t-1})y_{t-1}}{\sigma^3}, \tag{6}$$

$$\frac{\partial^2}{\partial\sigma^2} = \frac{T-1}{\sigma^2} - \frac{3\sum(y_t - \rho y_{t-1})}{\sigma^4}. \tag{7}$$

To form the information matrix I, evaluate the second derivatives at the maximum of the likelihood and change their signs:

$$I = \begin{pmatrix} \frac{\sum y_{t-1}^2}{\hat\sigma^2} & 0 \\ 0 & -\frac{T-1}{\hat\sigma^2} + 3\frac{\sum(y_t-\rho y_{t-1})^2}{\hat\sigma^4} \end{pmatrix} \tag{8}$$

In the above we used the FOC with respect to ρ which implies that the cross derivatives are equal to 0. Next notice that we can simplify the information using the formula for $\hat\sigma^2$ derived above:

$$I = \begin{pmatrix} \frac{\sum y_{t-1}^2}{\hat\sigma^2} & 0 \\ 0 & 2\frac{T-1}{\hat\sigma^2} \end{pmatrix} \tag{9}$$

The Cramer-Rao lower bound is the inverse of the information matrix, or:

$$I^{-1} = \frac{1}{\frac{\sum y_{t-1}^2}{\hat\sigma^2} 2\frac{T-1}{\hat\sigma^2}} \begin{pmatrix} 2\frac{T-1}{\hat\sigma^2} & 0 \\ 0 & \frac{\sum y_{t-1}^2}{\hat\sigma^2} \end{pmatrix} \tag{10}$$

Thus, we can see that:

$$var(\hat\rho) = \frac{\hat\sigma^2}{\sum y_{t-1}^2}, \tag{11}$$

$$var(\hat\sigma) = \frac{\hat\sigma^2}{2(T-1)}. \tag{12}$$

But these estimators again coincide with their OLS counterparts.