Online appendix to "Interpreting Shocks to the Relative Price of Investment with a Two-Sector Model" by Luca Guerrieri, Dale Henderson, and Jinill Kim

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This appendix provides additional details on the proof of Theorem 1 in the main body of the paper. We first fully describe the model to which Theorem 1 refers. Second, we derive the conditions for an equilibrium. Third, we use the conditions for an equilibrium to derive a set of steady-state conditions. Finally, we provide all the intermediate steps omitted from the main body of the paper to derive Equation 22 and Equation 28 in the paper, respectively transcribed as equations 50 and 70 in this appendix.

1 The MFP model

In period s, the representative household consumes C_s, supplies labor L_s, chooses next period's capital for the machinery sector, K_{Ms+1}, and for the non-machinery sector, K_{Ns+1}, as well as the borrowing level, B_s, so as to maximize the intertemporal utility function

$$\max_{Cs,I_s,K_{Ns+1},K_{Ms+1},B_s} E_t \sum_{s=t}^{\infty} \left[\tilde{\beta}_s \left((1-\eta)\log(C_s - \eta \bar{C}_{s-1}) - \frac{\chi_0}{1+\chi} V_s \left(L_s\right)^{1+\chi} \right) \right]. \tag{1}$$

The term $\tilde{\beta}_s$ denotes the household's time-varying discount factor, while η parameterizes external habit persistence in consumption. The parameter χ governs the household's labor supply elasticity, while χ_0 governs hours worked in the steady state. The household is subject to the labor supply shock V_s, which evolves according to an auto-regressive process

$$\log\left(V_s\right) = \rho_V \log\left(V_{s-1}\right) + \epsilon_{Vs}, \tag{2}$$

where log denotes the natural logarithm, ρ_V is the parameter governing the persistence of the auto-regressive process and ϵ_{Vs} is a stochastic innovation drawn from a Normal distribution with standard deviation σ_v. In turn, the discount factor is defined as $\tilde{\beta}_s = \frac{1}{\beta_{t-1}} \prod_{z=t-1}^{s-1} \beta_z$, with β_t evolving according to another auto-regressive process

$$\beta_t - \beta = \rho_\beta(\beta_t - \beta) + \epsilon_{\beta t}. \tag{3}$$

For the process above, ρ_β is the persistence parameter, ϵ_{Vs} is a stochastic innovation drawn from a Normal distribution with standard deviation σ_β, and β is the steady-state discount factor.

The household optimization problem is subject to the budget constraint

$$W_s L_s + R_{Ms} K_{Ms} + R_{Ns} K_{Ns} + \rho_{s-1} B_{s-1} = P_{Cs} C_s + P_{Is} I_s + B_s, \tag{4}$$

where W_s is the wage rate, R_{Ms} and R_{Ns}, are the rental rates for K_{Ms} and K_{Ns}, repectively, and ρ_s is the gross interest rate paid on previous period's borrowing. On the right-hand side of the constraint, P_{Cs} is the price of final consumption goods and P_{Is} is the price of final investment goods, I_s. The optimization problem is also subject to the law of motion for the accumulation of capital

$$K_{Ms+1} + K_{Ns+1} = (1 - \delta_M) K_{Ms} + (1 - \delta_N) K_{Ns} + I_s - \frac{\nu}{2} I_s \left(\frac{I_s}{I_{s-1}} - 1 \right)^2, \tag{5}$$

where δ_M and δ_N are the depreciation rates for K_{Ms} and K_{Ns}, respectively, andν parameterizes the adjustment costs for investment.

In each sector, perfectly competitive firms minimize production costs to meet demand subject to the technology constraint as reflected in the following Lagrangian problems:

$$\min_{K_{Ms}, L_{Ms}, P_{Ms}} R_{Ms} K_{Ms} + W_s L_{Ms} + P_{Ms}(Y_{Ms} - K_{Ms}^{\alpha_M} \left(A_{Ms} L_{Ms}\right)^{1-\alpha_M}), \tag{6}$$

$$\min_{K_{Ns}, L_{Ns}, P_{Ns}} R_{Ns} K_{Ns} + W_s L_{Ns} + P_{Ns}(Y_{Ns} - K_{Ns}^{\alpha_N} \left(A_{Ns} L_{Ns}\right)^{1-\alpha_N}), \tag{7}$$

where α_M and α_N denote the capital intensities in the production of M and N goods, respectively. The sectoral productivity levels A_{Ms} and A_{Ns}evolve according to the following stochastic processes:

$$A_{Ms} = A_{Ms-1} + \epsilon_{Ms} + \epsilon_{As}, \tag{8}$$

$$A_{Ms} = A_{Ms-1} + \epsilon_{As}, \tag{9}$$

where ϵ_{Ms} is a stochastic innovation, drawn from a Normal distribution with standard deviation σ_M, that is specific to productivty in sector M, and where ϵ_{As} is a stochastic innovation, drawn from a

Normal distribution with standard deviation σ_A, that is common to productivity in sectors M and N (i.e., sector-neutral).

Competitive final producers repackage the intermediate inputs to produce consumption and investment goods. Consumption producers minimize the cost of producing a desired level of consumption goods, split between private consumption C_s and government consumption G_{Cs}, by solving the following Lagrangian problem:

$$\min_{Y_{MCs}, Y_{NCs}, P_{Cs}} P_{Ns} Y_{NCs} + P_{Ms} Y_{MCs} - P_{Cs} \left[Y_{NCs}^{\alpha_{NC}} Y_{MCs}^{1-\alpha_{NC}} - (C_s + G_{Cs}) \right], \tag{10}$$

where α_{NC} governs the intensity of N-sector goods in the production of final consumption goods. In turn, government consumption follows a simple auto-regressive process:

$$G_{Cs} = \rho_{GC} G_{Cs} + \epsilon_{GCs}, \tag{11}$$

where the parameter ρ_{GC} governs the persistence of the shock process, and where ϵ_{GCs} is a stochastic innovation drawn from Normal distribution with standard deviations σ_{GC}. Investment producers solve the analogous problem:

$$\min_{Y_{MIs}, Y_{NIs}, P_{Is}} P_{Ms} Y_{MIs} + P_{Ns} Y_{MIs} - P_{Is} \left[Y_{NIs}^{\alpha_{NI}} Y_{MIs}^{1-\alpha_{NI}} - I_s \right], \tag{12}$$

with α_{NI} governing the intensity of N-sector goods in the production of final investment goods.

In addition to satisfying the first-order conditions for the optimization problems of households and firms, an equilibrium in the model has no borrowing (i.e., $B_s = 0 \forall s$), and is such that all factor markets and product markets clear. Accordingly,

$$Y_{Ms} = Y_{MCs} + Y_{MIs}, \tag{13}$$

$$Y_{Ns} = Y_{NCs} + Y_{NIs}, \tag{14}$$

$$L_s = L_{Ms} + L_{Ns}. \tag{15}$$

2 Necessary Conditions for an equilibrium

From the household's side, let λ_{Cs} be the Lagrange multiplier on the budget constraint and λ_{Ks} be the Lagrange multiplier on the capital accumulation equation.

From $\frac{\partial}{\partial C_s} = 0$

$$\frac{1-\eta}{C_s - \eta C_{s-1}} - \lambda_{Cs} P_{Cs} = 0. \tag{16}$$

From $\frac{\partial}{\partial I_s} = 0$

$$-\lambda_{Cs} P_{Is} - \lambda_{Ks} \left[1 - \frac{\nu}{2} \left(\frac{I_s}{I_{s-1}} - 1 \right)^2 - \nu \left(\frac{I_s}{I_{s-1}} - 1 \right) \frac{I_s}{I_{s-1}} \right] - \lambda_{Ks+1} \nu \left(\frac{I_{s+1}}{I_s} - 1 \right) \left(\frac{I_{s+1}}{I_s} \right)^2 = 0. \tag{17}$$

From $\frac{\partial}{\partial K_{Ns+1}} = 0$

$$\lambda_{Cs+1} \beta_s E_s R_{Ns+1} + \lambda_{Ks} - E_s \lambda_{Ks+1} \beta_t (1 - \delta_N) = 0. \tag{18}$$

From $\frac{\partial}{\partial K_{Ms+1}} = 0$

$$\lambda_{Cs+1} \beta_s E_s R_{Ms+1} + \lambda_{Ks} - E_s \lambda_{Ks+1} \beta_s (1 - \delta_M) = 0. \tag{19}$$

From $\frac{\partial}{\partial B_s} = 0$

$$-\lambda_{Cs} + \beta E_s \lambda_{Cs+1} \rho_s = 0. \tag{20}$$

From $\frac{\partial}{\partial L_s}$

$$-\chi_0 L_s^{\chi} V_s + \lambda_{Cs} W_s = 0. \tag{21}$$

From $\frac{\partial}{\partial \lambda_{Ks}}$

$$K_{Ms+1} + K_{Ns+1} = (1 - \delta_M) K_{Ms} + (1 - \delta_N) K_{Ns} + I_s. \tag{22}$$

From $\frac{\partial}{\partial \lambda_{Cs}}$

$$W_s L_s + R_{Ms} K_{Ms} + R_{Ns} K_{Ns} + \rho_{s-1} B_{s-1} = P_{Cs} C_s + P_{Is} I_s + B_s. \tag{23}$$

From the firms' problem using $\frac{\partial}{\partial K_{Ms}} = 0$, we get $R_{Ms} - P_{Ms} \alpha_M K_{Ms}^{\alpha_M - 1} (A_M L_{Ms})^{1-\alpha_M} = 0$, and rearranging

$$R_{Ms} - P_{Ms} \alpha_M \frac{Y_{Ms}}{K_{Ms}} = 0. \tag{24}$$

From $\frac{\partial}{\partial L_{Ms}} = 0$, we get $W_s - P_{Ms}\left(1-\alpha_M\right) K_{Ms}^{\alpha_M}\left(A_M L_{Ms}\right)^{-\alpha_M} A_M = 0$ and rearranging

$$W_s - P_{Ms}\left(1-\alpha_M\right)\frac{Y_{Ms}}{L_{Ms}} = 0. \tag{25}$$

From $\frac{\partial}{\partial P_{Ms}} = 0$

$$Y_{Ms} - K_{Ms}^{\alpha_M}\left(A_M L_{Ms}\right)^{1-\alpha_M}. \tag{26}$$

From $\frac{\partial}{\partial K_{Ns}} = 0$, we get $R_{Ns} - P_{Ns}\alpha_N K_{Ns}^{\alpha_N - 1}\left(A_N L_{Ns}\right)^{1-\alpha_N} = 0$ and rearranging

$$R_{Ns} - P_{Ns}\alpha_N \frac{Y_{Ns}}{K_{Ns}} = 0. \tag{27}$$

From $\frac{\partial}{\partial L_{Ns}} = 0$, we get $W_s - P_{Ns}\left(1-\alpha_N\right) K_{Ns}^{\alpha_N}\left(A_N L_{Ns}\right)^{-\alpha_N} A_N = 0$ and rearranging

$$W_s - P_{Ns}\left(1-\alpha_N\right)\frac{Y_{Ns}}{L_{Ns}} = 0. \tag{28}$$

From $\frac{\partial}{\partial P_{Ns}} = 0$

$$Y_{Ns} - K_{Ns}^{\alpha_N}\left(A_N L_{Ns}\right)^{1-\alpha_N}. \tag{29}$$

Next, consider the cost minimization problems for the final producers. $\min_{Y_{MCs}, Y_{NCs}, P_{Cs}} P_{Ns} Y_{NCs} + P_{Ms} Y_{MCs} - P_{Cs}\left(Y_{NCs}^{\alpha_{NC}} Y_{MCs}^{1-\alpha_{NC}} - C_s - G_{Cs}\right).$

From $\frac{\partial}{\partial Y_{NCs}} = 0$

$$P_{Ns} - P_{Cs}\alpha_{NC}\frac{C_s + G_{Cs}}{Y_{NCs}} = 0$$

and rearranging

$$Y_{NCs} = \alpha_{NC}\left(C_s + G_{Cs}\right)\frac{P_{Cs}}{P_{Ns}}. \tag{30}$$

From $\frac{\partial}{\partial Y_{MCs}} = 0$

$$Y_{MCs} = \left(1-\alpha_{NC}\right)\left(C_s + G_{Cs}\right)\frac{P_{Cs}}{P_{Ms}}. \tag{31}$$

Combining conditions 30and 31 with the $C_s + G_{Cs} = Y_{NCs}^{\alpha_{NC}} Y_{MCs}^{1-\alpha_{NC}}$,

$$C_s + G_{Cs} = \left(\alpha_{NC}\left(C_s + G_{Cs}\right)\frac{P_{Cs}}{P_{Ns}}\right)^{\alpha_{NC}}\left(\left(1-\alpha_{NC}\right)\left(C_s + G_{Cs}\right)\frac{P_{Cs}}{P_{Ms}}\right)^{1-\alpha_{NC}}$$

$$C_s + G_{Cs} = \left(C_s + G_{Cs}\right)P_{Cs}\left(\alpha_{NC}\frac{1}{P_{Ns}}\right)^{\alpha_{NC}}\left(\left(1-\alpha_{NC}\right)\frac{1}{P_{Ms}}\right)^{1-\alpha_{NC}}$$

$$1 = P_{Cs}\left(\alpha_{NC}\frac{1}{P_{Ns}}\right)^{\alpha_{NC}}\left(\left(1-\alpha_{NC}\right)\frac{1}{P_{Ms}}\right)^{1-\alpha_{NC}}$$

$$P_{Cs} = \left(\frac{P_{Ns}}{\alpha_{NC}}\right)^{\alpha_{NC}}\left(\frac{P_{Ms}}{1-\alpha_{NC}}\right)^{1-\alpha_{NC}}.$$

From $\frac{\partial}{\partial Y_{NIs}} = 0$

$$Y_{NIs} = \alpha_{NI}\left(I_s + G_{Is}\right)\frac{P_{Is}}{P_{Ns}}. \tag{32}$$

From $\frac{\partial}{\partial Y_{MIs}} = 0$

$$Y_{MIs} = \left(1-\alpha_{NI}\right)\left(I_s + G_{Is}\right)\frac{P_{Is}}{P_{Ms}}. \tag{33}$$

And analogously to P_{Cs}, derived above, P_{Is} is given by

$$P_{Is} = \left(\frac{P_{Ns}}{\alpha_{NI}}\right)^{\alpha_{NI}}\left(\frac{P_{Ms}}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}.$$

In addition to these first-order conditions, the labor market and product markets must clear:

$$L_s = L_{Ms} + L_{Ns}, \tag{34}$$

$$Y_{Ms} = Y_{MCs} + Y_{MIs}, \tag{35}$$

$$Y_{Ns} = Y_{NCs} + Y_{NIs}. \tag{36}$$

Finally, choose units by setting

$$P_{Ns} = 1, \tag{37}$$

and include all the stochastic processes described in the previous section:

$$\log\left(V_s\right) = \rho_V \log\left(V_{s-1}\right) + \epsilon_{Vs},$$

$$A_{Ms} = A_{Ms-1} + \epsilon_{Ms} + \epsilon_s,$$

$$A_{Ms} = A_{Ms-1} + \epsilon_s,$$

$$G_{Cs} = \rho_{GC} G_{Cs} + \epsilon_{GCs},$$

$$G_{Is} = \rho_{GI} G_{Is} + \epsilon_{GIs}.$$

3 Derivation of some steady-state restrictions

Equation I)

Work on $\frac{\partial}{\partial K_{Ns}} = 0$, from which we had

$$\lambda_{Cs+1} \beta E_s R_{Ns+1} + \lambda_{Ks} - E_s \lambda_{Ks+1} \beta (1 - \delta_N) = 0.$$

From $\frac{1}{C_s} - \lambda_{Cs} P_{Cs} = 0$,

$$\frac{1}{P_C C} = \lambda_C.$$

Furthermore, with $-\lambda_{Cs} P_{Is} = \lambda_{Ks}$, which can be expressed as $-\frac{P_{Is}}{P_{Cs} C_s} = \lambda_K$ one obtains:

$$\beta \frac{R_N}{P_C C} - \frac{P_I}{P_C C} + \beta \frac{P_I}{P_C C}(1 - \delta_N) = 0.$$

Equation II)

Combining $\frac{\partial}{\partial K_{Ns}} = 0$ and $\frac{\partial}{\partial K_{Ms}} = 0$

$$\lambda_{Cs+1} \beta E_s R_{Ns+1} + \lambda_{Ks} - E_s \lambda_{Ks+1} \beta (1 - \delta_N) = 0,$$

$$\lambda_{Cs+1} \beta E_s R_{Ms+1} + \lambda_{Ks} - E_s \lambda_{Ks+1} \beta (1 - \delta_M) = 0.$$

Turn to steady state and divide by λ_C to obtain:

$$\beta R_N = -\frac{\lambda_K}{\lambda_C} + \frac{\lambda_K}{\lambda_C} \beta (1 - \delta_N).$$

$$\beta R_M = -\frac{\lambda_K}{\lambda_C} + \frac{\lambda_K}{\lambda_C} \beta (1 - \delta_M).$$

Collecting terms

$$\beta R_N = \frac{\lambda_K}{\lambda_C} \left(-1 + \beta(1 - \delta_N)\right),$$

$$\beta R_M = \frac{\lambda_K}{\lambda_C} \left(-1 + \beta(1 - \delta_M)\right).$$

Dividing the two

$$\frac{R_N}{R_M} = \frac{1 - \beta(1 - \delta_N)}{1 - \beta(1 - \delta_M)}.$$

Equation III)

From the firms' problem, using $\frac{\partial}{\partial K_{Ms}} = 0$

$$R_M = P_M \alpha_M \frac{Y_M}{K_M}.$$

Equation IV)

From the firms' problem, using $\frac{\partial}{\partial L_{Ms}} = 0$

$$W = P_M \left(1 - \alpha_M\right) \frac{Y_M}{L_M}.$$

Equation V)

From the firms' problem, using $\frac{\partial}{\partial K_{Ns}} = 0$

$$R_N = P_N \alpha_N \frac{Y_N}{K_N}.$$

Equation VI)

From the firms' problem, using $\frac{\partial}{\partial L_{Ms}} = 0$

$$W = P_N \left(1 - \alpha_N\right) \frac{Y_N}{L_N}.$$

Equation VII)

Using the production technology for sector M,

$$Y_M = K_M^{\alpha_M} \left(A_M L_M\right)^{1-\alpha_M}.$$

Equation VIII)

Using the production technology for sectory N,

$$Y_N = K_N^{\alpha_N} \left(A_N L_N\right)^{1-\alpha_N}.$$

Equation IX)

From the problem of final consumption producers,

$$Y_{NC} = \alpha_{NC} C \frac{P_C}{P_N}.$$

Equation X)

From the problem of final consumption producers,

$$P_{Cs} = \left(\frac{P_{Ns}}{\alpha_{NC}}\right)^{\alpha_{NC}} \left(\frac{P_{Ms}}{1-\alpha_{NC}}\right)^{1-\alpha_{NC}}.$$

Equation XI)

From the problem of final investment producers,

$$Y_{NIs} = \alpha_{NI} I_s \frac{P_{Is}}{P_{Ns}}.$$

Equation XII)

$$P_I = \left(\frac{P_{Ns}}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_{Ms}}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}.$$

Equation XIII)

From market clearing

$$L_M + L_N = L.$$

Equation XIV)

From market clearing

$$Y_M = Y_{MC} + Y_{MI}.$$

Equation XV)

$$Y_N = Y_{NC} + Y_{NI}.$$

Equation XVI)

Using the capital accumulation equation, $K_{Ms+1} + K_{Ns+1} = (1-\delta_M)K_{Ms} + (1-\delta_N)K_{Ns} + I_s$, with complete specialization

$$\delta_M K_M + \delta_N K_N = I.$$

Equation XVII)

$$\rho = \frac{1}{\beta}.$$

Equation XVIII)

$$-\chi_0 L^{\chi} + \lambda_C W = 0.$$

Equation XIX)

Normalizing units:

$$P_N = 1.$$

4 Proof of Theorem 1: Part 1, The Long-Run Response of Relative Prices

Combining equations X) and XIX)

$$P_C = \left(\frac{1}{\alpha_{NC}}\right)^{\alpha_{NC}} \left(\frac{P_M}{1-\alpha_{NC}}\right)^{1-\alpha_{NC}}.$$

Combining equations XII) and XIX)

$$P_I = \left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}.$$

From I), multiplying both sides by $P_C C$

$$\beta R_N - P_I + \beta P_I(1-\delta_N) = 0, \tag{38}$$

and rearranging

$$R_N = P_I \left(\frac{1}{\beta} - (1 - \delta_N) \right). \tag{39}$$

Combing the equation above with II) $\frac{R_N}{R_M} = \frac{1-\beta(1-\delta_N)}{1-\beta(1-\delta_M)}$.

$$\frac{R_N}{R_M} = \frac{1-\beta(1-\delta_N)}{1-\beta(1-\delta_M)}.$$

$$\frac{1}{\beta} P_I \left(1 - \beta(1-\delta_N)\right) = \frac{1-\beta(1-\delta_N)}{1-\beta(1-\delta_M)} R_M,$$

$$R_M = P_I \left(\frac{1}{\beta} - (1 - \delta_M) \right). \tag{40}$$

Solve VII) for K_M

$$K_M = \left(\frac{Y_M}{(A_M L_M)^{1-\alpha_M}} \right)^{\frac{1}{\alpha_M}} \tag{41}$$

and substitute it into III) to yield:

$$R_M = P_M \alpha_M \frac{Y_M}{\left(\frac{Y_M}{(A_M L_M)^{1-\alpha_M}} \right)^{\frac{1}{\alpha_M}}},$$

which simplifies to

$$\frac{\left(\frac{Y_M}{(A_M L_M)^{1-\alpha_M}} \right)^{\frac{1}{\alpha_M}}}{Y_M} = \alpha_M \frac{P_M}{R_M},$$

$$\frac{Y_M^{\frac{1-\alpha_M}{\alpha_M}}}{(A_M L_M)^{\frac{1-\alpha_M}{\alpha_M}}} = \alpha_M \frac{P_M}{R_M},$$

$$\frac{Y_M}{L_M} = A_M \left(\alpha_M \frac{P_M}{R_M}\right)^{\frac{\alpha_M}{1-\alpha_M}}. \tag{42}$$

Analogously from V) and VIII), we obtain:

$$\frac{Y_N}{L_N} = A_N \left(\alpha_N \frac{P_N}{R_N}\right)^{\frac{\alpha_N}{1-\alpha_N}}. \tag{43}$$

Next, combine IV) and VI) to yield:

$$\frac{P_M}{P_N} = \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{Y_N}{L_N} \frac{L_M}{Y_M}. \tag{44}$$

Substituting equations 42, and 43 into equation 44, one can solve for $\frac{P_M}{P_N}$ in terms of parameters and the levels of sector-specific technology A_M and A_N:

$$\frac{P_M}{P_N} = \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{A_N \left(\alpha_N \frac{P_N}{P_I\left(\frac{1}{\beta}-(1-\delta_N)\right)}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{A_M \left(\alpha_M \frac{P_M}{\frac{1}{\beta}P_I(1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}}. \tag{45}$$

But remembering that $P_I = \left(\frac{P_N}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}$

$$\frac{P_M}{P_N} = \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{A_N \left(\alpha_N \frac{P_N}{\left(\frac{P_N}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}} \left(\frac{1}{\beta}-(1-\delta_N)\right)}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{A_M \left(\alpha_M \frac{P_M}{\frac{1}{\beta}\left(\frac{P_N}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}} (1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}}. \tag{46}$$

$$\frac{P_M}{P_N} = \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{A_N \left(\alpha_N \frac{\left(\frac{P_N}{P_M}\right)^{1-\alpha_{NI}}}{\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}} \left(\frac{1}{\beta}-(1-\delta_N)\right)}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{A_M \left(\alpha_M \frac{\left(\frac{P_M}{P_N}\right)^{\alpha_{NI}}}{\frac{1}{\beta}\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}} (1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}}. \tag{47}$$

$$\frac{P_M}{P_N}=\frac{(1-\alpha_N)}{(1-\alpha_M)}\frac{A_N\left(\alpha_N\frac{1}{\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}}\left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}\left(\frac{1}{\beta}-(1-\delta_N)\right)}\right)^{\frac{\alpha_N}{1-\alpha_N}}\left(\frac{P_N}{P_M}\right)^{\frac{(1-\alpha_{NI})\alpha_N}{1-\alpha_N}}\left(\frac{P_N}{P_M}\right)^{\frac{\alpha_{NI}\alpha_M}{1-\alpha_M}}}{A_M\left(\alpha_M\frac{1}{\frac{1}{\beta}\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}}\left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}(1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}} \tag{48}$$

$$\frac{P_M}{P_N}=\frac{(1-\alpha_N)}{(1-\alpha_M)}\frac{A_N\left(\alpha_N\frac{1}{\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}}\left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}\left(\frac{1}{\beta}-(1-\delta_N)\right)}\right)^{\frac{\alpha_N}{1-\alpha_N}}\left(\frac{P_N}{P_M}\right)^{\frac{(1-\alpha_M)(1-\alpha_{NI})\alpha_N+(1-\alpha_N)\alpha_{NI}\alpha_M}{(1-\alpha_N)(1-\alpha_M)}}}{A_M\left(\alpha_M\frac{1}{\frac{1}{\beta}\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}}\left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}(1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}} \tag{49}$$

$$\frac{P_M}{P_N}=\left(\psi\frac{A_N}{A_M}\right)^{\frac{(1-\alpha_N)(1-\alpha_M)}{(1-\alpha_N)(1-\alpha_M)+(1-\alpha_M)(1-\alpha_{NI})\alpha_N+(1-\alpha_N)\alpha_{NI}\alpha_M}},\quad \text{where} \tag{50}$$

$$\psi=\left(\frac{(1-\alpha_N)}{(1-\alpha_M)}\frac{\left(\alpha_N\frac{1}{\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}}\left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}\left(\frac{1}{\beta}-(1-\delta_N)\right)}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{\left(\alpha_M\frac{1}{\frac{1}{\beta}\left(\frac{1}{\alpha_{NI}}\right)^{\alpha_{NI}}\left(\frac{1}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}(1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}}\right).$$

Thus, equiproportionate changes in technology in the two production sectors M and N will not affect relative prices. Variation in relative prices at the sectoral level is a precondition for variation in relative prices at the level of final goods. Thus, the result derived here extends to the model in the main body of the paper with incomplete sectoral specialization in the assembly of consumption and investment goods, as reflected in the numerical simulations.

5 Proof of Theorem 1: Part 2, The Long-Run Response of Labor Productivity

Define labor productivity (at constant prices) as:

$$\frac{Y_{Mt}+Y_{Nt}}{L}=\frac{Y_{Mt}}{L_{Mt}}\frac{L_{Mt}}{L}+\frac{Y_{Nt}}{L_{Nt}}\frac{L_{Nt}}{L}. \tag{51}$$

First work on obtaining $\frac{Y_N}{Y}$ and $\frac{Y_M}{Y}$ in terms of parameters and the relative technology level, only.

Using V), $R_N = P_N \alpha_N \frac{Y_N}{K_N}$, and 39, $R_N = P_I \left(\frac{1}{\beta} - (1-\delta_N)\right)$, derive

$$\frac{K_N}{Y_N} = \frac{\alpha_N}{\left(\frac{1}{\beta} - (1-\delta_N)\right)} \frac{P_N}{P_I}. \tag{52}$$

To see this result start from:

$$\frac{Y_N}{K_N} = \frac{R_N}{P_N \alpha_N}$$

$$\frac{Y_N}{K_N} = \frac{P_I \left(\frac{1}{\beta} - (1-\delta_N)\right)}{P_N \alpha_N}$$

and rearranging:

$$\frac{K_N}{Y_N} = \frac{P_N \alpha_N}{P_I \left(\frac{1}{\beta} - (1-\delta_N)\right)}$$

And similarly, using III), and 40, one can obtain

$$\frac{K_M}{Y_M} = \frac{P_M \alpha_M}{P_I \left(\frac{1}{\beta} - (1-\delta_M)\right)}, \tag{53}$$

$$Y = P_c C + P_I I.$$

Define the saving rate as $S = \frac{P_I * I}{Y}$. And define

$$Y = P_N Y_N + P_M Y_M$$

And from the resource constraints:

$$Y_N = Y_{NI} + Y_{NC} \tag{54}$$

$$Y_M = Y_{MI} + Y_{MC} \tag{55}$$

But we can express Y_{NI} and the other inputs in terms of relative prices using the demand equations:

$$Y_N = \alpha_{NI} I \frac{P_I}{P_N} + \alpha_{NC} C \frac{P_C}{P_N} \tag{56}$$

$$Y_M = (1-\alpha_{NI})\, I \frac{P_I}{P_M} + (1-\alpha_{NC})\, C \frac{P_C}{P_M} \tag{57}$$

Using $S = \frac{P_I * I}{Y}$.

$$\frac{Y_N}{Y} = \alpha_{NI} \frac{S}{P_N} + \alpha_{NC} \frac{(1-S)}{P_N} \tag{58}$$

$$\frac{Y_M}{Y} = (1-\alpha_{NI}) \frac{S}{P_M} + (1-\alpha_{NC}) \frac{(1-S)}{P_M} \tag{59}$$

$$\delta_N \frac{K_N}{Y_N} \frac{Y_N}{Y} P_I + \delta_M \frac{K_M}{Y_M} \frac{Y_M}{Y} P_I = S$$

Substitute S from the third equation into the first two.

$$\frac{Y_N}{Y} = \alpha_{NI} \left(\delta_N \frac{K_N}{Y_N} \frac{Y_N}{Y} P_I + \delta_M \frac{K_M}{Y_M} \frac{Y_M}{Y} P_I \right) \frac{1}{P_N} + \alpha_{NC} \left(1 - \delta_N \frac{K_N}{Y_N} \frac{Y_N}{Y} P_I - \delta_M \frac{K_M}{Y_M} \frac{Y_M}{Y} P_I \right) \frac{1}{P_N} \tag{60}$$

$$\frac{Y_M}{Y} = (1-\alpha_{NI}) \left(\delta_N \frac{K_N}{Y_N} \frac{Y_N}{Y} P_I + \delta_M \frac{K_M}{Y_M} \frac{Y_M}{Y} P_I \right) \frac{1}{P_{Ms}} + (1-\alpha_{NC}) \left(1 - \delta_N \frac{K_N}{Y_N} \frac{Y_N}{Y} P_I - \delta_M \frac{K_M}{Y_M} \frac{Y_M}{Y} P_I \right) \frac{1}{P_M} \tag{61}$$

Use the first equation above to solve for $\frac{Y_N}{Y}$.

$$\begin{aligned} &\tfrac{Y_N}{Y} - \alpha_{NI} \left(\delta_N \tfrac{K_N}{Y_N} \tfrac{Y_N}{Y} P_I \right) \tfrac{1}{P_{Ns}} + \alpha_{NC} \left(\delta_N \tfrac{K_N}{Y_N} \tfrac{Y_N}{Y} P_I \right) \tfrac{1}{P_N} \\ &= \alpha_{NI} \left(\delta_M \tfrac{K_M}{Y_M} \tfrac{Y_M}{Y} P_I \right) \tfrac{1}{P_N} + \alpha_{NC} \left(1 - \delta_M \tfrac{K_M}{Y_M} \tfrac{Y_M}{Y} P_I \right) \tfrac{1}{P_{Ns}} \quad , \end{aligned} \tag{62}$$

$$\begin{aligned} &\left[1 - \alpha_{NI} \left(\delta_N \tfrac{K_N}{Y_N} P_I \right) \tfrac{1}{P_{Ns}} + \alpha_{NC} \left(\delta_N \tfrac{K_N}{Y_N} P_I \right) \tfrac{1}{P_{Ns}} \right] \tfrac{Y_N}{Y} \\ &= \left[\alpha_{NI} \left(\delta_M \tfrac{K_M}{Y_M} P_I \right) \tfrac{1}{P_{Ns}} - \alpha_{NC} \left(\delta_M \tfrac{K_M}{Y_M} P_I \right) \tfrac{1}{P_{Ns}} \right] \tfrac{Y_M}{Y} + \alpha_{NC} \tfrac{1}{P_{Ns}} \quad , \end{aligned} \tag{63}$$

$$\frac{Y_N}{Y} = \frac{\left[\alpha_{NI} \left(\delta_M \tfrac{K_M}{Y_M} P_I \right) \tfrac{1}{P_{Ns}} - \alpha_{NC} \left(\delta_M \tfrac{K_M}{Y_M} P_I \right) \tfrac{1}{P_{Ns}} \right] \tfrac{Y_M}{Y} + \alpha_{NC} \tfrac{1}{P_{Ns}}}{\left[1 - \alpha_{NI} \left(\delta_N \tfrac{K_N}{Y_N} P_I \right) \tfrac{1}{P_N} + \alpha_{NC} \left(\delta_N \tfrac{K_N}{Y_N} P_I \right) \tfrac{1}{P_N} \right]}. \tag{64}$$

Re-write $\frac{Y_N}{Y}$ as $\frac{A \frac{Y_M}{Y} + B}{C}$. Working to simplify equation 61

$$\begin{aligned} &\tfrac{Y_M}{Y} - (1-\alpha_{NI}) \left(\delta_M \tfrac{K_M}{Y_M} \tfrac{Y_M}{Y} P_I \right) \tfrac{1}{P_M} + (1-\alpha_{NC}) \left(\delta_M \tfrac{K_M}{Y_M} \tfrac{Y_M}{Y} P_I \right) \tfrac{1}{P_M} \\ &= (1-\alpha_{NI}) \left(\delta_N \tfrac{K_N}{Y_N} \tfrac{Y_N}{Y} P_I \right) \tfrac{1}{P_{Ms}} + (1-\alpha_{NC}) \left(1 - \delta_N \tfrac{K_N}{Y_N} \tfrac{Y_N}{Y} P_I \right) \tfrac{1}{P_M} \quad , \end{aligned}$$

$$\left[1-(1-\alpha_{NI})\left(\delta_M\frac{K_M}{Y_M}P_I\right)\frac{1}{P_M}+(1-\alpha_{NC})\left(\delta_M\frac{K_M}{Y_M}P_I\right)\frac{1}{P_M}\right]\frac{Y_M}{Y}$$
$$=\left[(1-\alpha_{NI})\left(\delta_N\frac{K_N}{Y_N}P_I\right)\frac{1}{P_M}-(1-\alpha_{NC})\left(\delta_N\frac{K_N}{Y_N}P_I\right)\frac{1}{P_M}\right]\frac{Y_N}{Y}+(1-\alpha_{NC})\frac{1}{P_M}\quad .$$

We already know that $\frac{K_M}{Y_M}$ is a function of parameters and relative technology, as is every other term in the equation above, except $\frac{Y_M}{Y}$. Re-write $\frac{Y_M}{Y}$ as

$$D\frac{Y_M}{Y}=E\frac{Y_N}{Y}+F.$$

Substituting $\frac{Y_N}{Y}=\frac{A\frac{Y_M}{Y}+B}{C}$ into the above

$$D\frac{Y_M}{Y}=E\frac{A\frac{Y_M}{Y}+B}{C}+F,$$

and solving for $\frac{Y_M}{Y}$

$$CD\frac{Y_M}{Y}=E\left(A\frac{Y_M}{Y}+B\right)+CF,$$

$$\frac{Y_M}{Y}=\frac{EB+CF}{CD-EA}.$$

Substituting $\frac{Y_M}{Y}=\frac{EB+CF}{CD-EA}$ back into $\frac{Y_N}{Y}=\frac{A\frac{Y_M}{Y}+B}{C}$

$$\frac{Y_N}{Y}=\frac{A\frac{EB+CF}{CD-EA}+B}{C},$$

$$\frac{Y_N}{Y}=\frac{A\frac{EB+CF}{CD-EA}+B}{C}=\frac{ACF+BCD}{C\left(CD-EA\right)}.$$

Dividing $\frac{Y_N}{Y}$ by $\frac{Y_M}{Y}$ one can see that

$$\frac{Y_N}{Y_M}=\frac{ACF+BCD}{BCE+C^2F},$$

which is a function of parameters and relative technology only. Combining IV, $W=P_M\left(1-\alpha_M\right)\frac{Y_M}{L_M}$, VI, $W=P_N\left(1-\alpha_N\right)\frac{Y_N}{L_N}$, and XIII, one obtains

$$\frac{L_M}{L_M+L_N}=\frac{P_M\left(1-\alpha_M\right)\frac{Y_M}{W}}{P_M\left(1-\alpha_M\right)\frac{Y_M}{W}+P_N\left(1-\alpha_N\right)\frac{Y_N}{W}}, \tag{65}$$

which further simplifies to

$$\frac{L_M}{L} = \frac{(1-\alpha_M)\frac{P_M}{P_N}}{(1-\alpha_M)\frac{P_M}{P_N} + (1-\alpha_N)\frac{Y_N}{Y_M}}. \tag{66}$$

And using the resource constraint $L_N + L_M = L$ one more time, one can see that

$$\frac{L_N}{L} = \frac{(1-\alpha_N)\frac{Y_N}{Y_M}}{(1-\alpha_M)\frac{P_M}{P_N} + (1-\alpha_N)\frac{Y_N}{Y_M}}. \tag{67}$$

Next work on $\frac{Y_M}{L_M}$ and on $\frac{Y_N}{L_N}$. Combining equations 42, $\frac{Y_M}{L_M} = A_M \left(\alpha_M \frac{P_M}{R_M}\right)^{\frac{\alpha_M}{1-\alpha_M}}$ and 40, $R_M = P_I\left(\frac{1}{\beta} - (1-\delta_M)\right)$, yields:

$$\frac{Y_M}{L_M} = A_M \left(\frac{\alpha_M}{\left(\frac{1}{\beta} - (1-\delta_M)\right)} \frac{P_M}{P_I}\right)^{\frac{\alpha_M}{1-\alpha_M}}.$$

Using $P_I = \left(\frac{P_N}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}$

$$\frac{Y_M}{L_M} = A_M \left(\frac{\alpha_M}{\left(\frac{1}{\beta} - (1-\delta_M)\right)} \frac{P_M}{\left(\frac{P_N}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}}\right)^{\frac{\alpha_M}{1-\alpha_M}}. \tag{68}$$

And similarly for $\frac{Y_{Nt}}{L_N}$, using 43 with equation 39

$$\frac{Y_N}{L_N} = A_N \left(\frac{\alpha_N}{\left(\frac{1}{\beta} - (1-\delta_N)\right)} \frac{P_N}{P_I}\right)^{\frac{\alpha_N}{1-\alpha_N}}.$$

Using $P_I = \left(\frac{P_N}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}$, again

$$\frac{Y_N}{L_N} = A_N \left(\frac{\alpha_N}{\left(\frac{1}{\beta} - (1-\delta_N)\right)} \frac{P_N}{\left(\frac{P_N}{\alpha_{NI}}\right)^{\alpha_{NI}} \left(\frac{P_M}{1-\alpha_{NI}}\right)^{1-\alpha_{NI}}}\right)^{\frac{\alpha_N}{1-\alpha_N}}. \tag{69}$$

Summing up, consider that $\frac{Y_M+Y_N}{L} = \frac{Y_M}{L_M}\frac{L_M}{L} + \frac{Y_N}{L_N}\frac{L_N}{L}$. Notice that from equations 68, 66, 69, and 67, the terms $\frac{Y_M}{L_M}$, $\frac{L_M}{L}$, $\frac{Y_N}{L_N}$, and $\frac{L_N}{L}$ are functions of parameters and relative as well as neutral technology. Thus, labor productivity, $\frac{Y_M+Y_N}{L}$, will also be a function of the same terms. Notice also that Fisher defined aggregate labor productivity in terms of consumption units (i.e., $\frac{Y_{Mt}}{L_{Mt}}\frac{L_{Mt}}{L}\frac{P_M}{P_N}+\frac{Y_N}{L_N}\frac{L_N}{L}$) rather than at constant prices. Even under that alternative aggregation, labor productivity remains a log-linear function of both shocks. Accordingly, taken together with $\frac{Y_M+Y_N}{L}=\frac{Y_M}{L_M}\frac{L_M}{L}+\frac{Y_N}{L_N}\frac{L_N}{L}$, equations 68, 66, 69, and 67 prove Theorem 1.

6 Complete specialization

If consumption is produced with only inputs from the N sector and investment is produced with only input from the M sector, in other words, under complete specialization ($\alpha_{NC}=1$,$\alpha_{NI}=0$), equations 50, derived in Section 4, and equations 68, 66, 69 and 67, derived in Section 5, simplify further. Equation 50 becomes

$$\frac{P_M}{P_N}=\left(\psi\frac{A_N}{A_M}\right)^{1-\alpha_N},\quad \text{where } \psi=\left(\frac{(1-\alpha_N)}{(1-\alpha_M)}\frac{\left(\alpha_N\frac{1}{\frac{1}{\beta}(1-\beta(1-\delta_N))}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{\left(\alpha_M\frac{1}{\frac{1}{\beta}(1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}}\right).$$

Furthermore, labor productivity can be expressed as

$$\begin{aligned}\frac{Y_M+Y_N}{L} &= \frac{Y_M}{L_M}\frac{L_M}{L}+\frac{Y_N}{L_N}\frac{L_N}{L}= \\ & A_M\left(\frac{\alpha_M}{(1-\beta(1-\delta_M))}\right)^{\frac{\alpha_M}{1-\alpha_M}}\frac{(1-\alpha_M)}{(1-\alpha_M)+(1-\alpha_N)\phi} \\ & +A_M^{\alpha_N}A_N^{1-\alpha_N}\left(\frac{\alpha_N}{\psi^{1-\alpha_N}(1-\beta(1-\delta_N))}\right)^{\frac{\alpha_N}{1-\alpha_N}}\frac{(1-\alpha_N)\phi}{(1-\alpha_M)+(1-\alpha_N)\phi},\end{aligned} \tag{70}$$

where $\phi=\left(\frac{1-\delta_M\frac{\alpha_M}{(1-\beta(1-\delta_M))}}{\delta_N\frac{\alpha_N}{(1-\beta(1-\delta_N))}}\right)$. This result can be seen from the fact that, in turn, equations 68, 66, 69 and 67 simplify, respectively, to

$$\frac{Y_M}{L_M}=A_M\left(\alpha_M\frac{P_M}{R_M}\right)^{\frac{\alpha_M}{1-\alpha_M}},$$

$$\frac{L_M}{L}=\frac{(1-\alpha_M)}{(1-\alpha_M)+(1-\alpha_N)\phi},$$

$$\frac{Y_N}{L_N}=A_N\left(\alpha_N\frac{P_N}{R_N}\right)^{\frac{\alpha_N}{1-\alpha_N}},$$

$$\frac{L_N}{L} = \frac{(1-\alpha_N)\phi}{(1-\alpha_M) + (1-\alpha_N)\phi}.$$